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GLOBAL $W_p^{1,2}$ ESTIMATES FOR NONDIVERGENCE PARABOLIC OPERATORS WITH POTENTIALS SATISFYING A REVERSE HÖLDER CONDITION

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Abstract. In this article, we first give the L^p boundedness of the operator D^2L^{-1} with BMO coefficients and a potential V satisfying an appropriate reverse Hölder condition, then obtain global $W_p^{1,2}$ estimates for the nondivergence parabolic operator L with VMO coefficients and a potential V satisfying an appropriate reverse Hölder condition.

1. Introduction

Throughout this paper we will use $x', y', \ldots,$ to indicate points in $\mathbb{R}^{n+1}$ and $x, y, \ldots,$ for points in $\mathbb{R}^n$ corresponding to the first n coordinates, i.e., we will write $x' = (x,t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1}$. We denote by $B(x,r)$ the ball of center x and radius r in $\mathbb{R}^n$.

Let L be the linear, second-order parabolic operator of the form

$$Lu(x') \equiv Au(x') + V(x)u(x') \equiv u_t(x') - a_{ij}(x')u_{x_ix_j}(x') + V(x)u(x'),$$

where $x' \in \mathbb{R}^{n+1}$. We assume that the principal part of the operator is bounded, symmetric and uniformly elliptic, i.e.,

(1.1)
$$a_{ij}(x') \in L^\infty(\mathbb{R}^{n+1}),\ a_{ij}(x') = a_{ji}(x') \text{ and } \varsigma|\xi'|^2 \le \sum_{i,j=1}^n a_{ij}(x')\xi'_i\xi'_j \le \frac{1}{\varsigma}|\xi'|^2$$

for $i, j = 1, 2, \ldots, n$ and some $\varsigma > 0$ and for every $x' \in \mathbb{R}^{n+1}$ and $\xi \in \mathbb{R}^n$.

In this paper, we always assume that

$$a_{ij}(x') \in BMO(\mathbb{R}^{n+1}) \tag{1.2}$$

which means that for parabolic BMO spaces and $\| a_{ij} \|_*$ stands for the BMO seminorm (see the definitions in the next section).

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We also always suppose V is not identically zero and

$$V \in B_q \quad \text{for some} \quad q \geq \frac{n}{2}, \tag{1.3}$$

which by definition means that $V \in L^q_{loc}(\mathbb{R}^n)$, $V \geq 0$ and there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(x)^q dx\right)^{\frac{1}{q}} \leq C\left(\frac{1}{|B|}\int_B V(x)dx\right) \tag{1.4}$$

holds for every ball B in $\mathbb{R}^n$.

Note that the B_q class has the following properties: if $V \in B_q$ for some $q > 1$, then there exists an $\epsilon > 0$, which depends only on n and the constant C in (1.4), such that $V \in B_{q+\epsilon}$. Clearly, V is a Muckenhoupt A_∞ weight [11, 15]. $V(y)dy$ is a doubling measure (see e.g. in [15]), that is, there exist positive constants α and C such that

$$\int_{B(x,2r)} V(y)dy \leq C2^\alpha \int_{B(x,r)} V(y)dy. \tag{1.5}$$

We define parabolic balls of center $x' = (x,t) \in \mathbb{R}^{n+1}$ and radius r by $Q(x',r) = \{y' = (y,s) \in \mathbb{R}^{n+1} : |x-y| < r, |t-s| < r^2\}$. Also by $\beta Q(x',r)$ we will mean the ball having the same center as $Q(x',r)$ and radius βr.

$$a_{ij} \in VMO(\mathbb{R}^{n+1}) \tag{1.6}$$

which means that for $i,j = 1,2,\ldots,n$, $a_{ij}(x') \in BMO(\mathbb{R}^{n+1})$ and

$$\eta_{ij}(r) = \sup_{\rho \leq r}\left(\frac{1}{|I_\rho|}\int_{I_\rho} |a_{ij}(y') - a_{ij}^{I_\rho}|dy'\right)$$

(which is finite for every r since a_{ij} is bounded) vanishes for $r \to 0^+$. Here I_ρ ranges over the class of parabolic balls in $\mathbb{R}^{n+1}$ of radius ρ and $a_{ij}^{I_\rho} = \frac{1}{|I_\rho|}\int_{I_\rho} a_{ij}(y')dy'$.

Recently, many people are interested in the global $W^{2,p}$ estimates and L^p boundedness of the elliptic operators and parabolic operators; see [2, 4–7, 10, 13, 14, 16, 18]. In particularly, under the assumption $V \in B_q$, Shen [14] proved the L^p boundedness of $D^2(-\Delta + V)^{-1}$ on $\mathbb{R}^n$, where $V = V(x)$ belongs to some reverse Hölder class. Then, Bramanti, Brandolini and Harboure [1] gave global $W^{2,p}$ estimates for elliptic operators $-a_{ij}(x)u_{x_ix_j} + V(x)u(x)$ with VMO coefficients and a potential V satisfying an appropriate reverse Hölder condition, see also [18]. Recently, the authors [12] obtained global $W^{2,p}$ estimates for elliptic operators with divergence and nondivergence forms with more general VMO coefficients and a potential V satisfying an appropriate reverse Hölder condition. On the parabolic operator case, A. Carbonaro, G. Metafune, C. Spina [4] got the L^p estimates of the parabolic operator $D^2(\partial_t - \Delta + V)^{-1}$ where $V = V(x,t)$ is a nonnegative potential which belongs to the Parabolic Reverse Hölder class; see also [8]. Later, Tang and Han [17] studied the L^p boundedness of other parabolic Schrödinger type operators. Recently, Tang

[16] obtained weighted L^p solvability for parabolic equations with partially BMO coefficients and nonpotentials.

Inspired by the above results, in this paper, we consider the global estimates for the parabolic operator L with appropriate assumptions.

2. Some preliminaries and notations

Let's now endow $\mathbb{R}^{n+1}$ withe the following parabolic metric. Define the parabolic distance

$$d((x,t),(y,s)) = (|x-y|^2 + |t-s|)^{\frac{1}{2}}$$

for any $(x,t),(y,s) \in \mathbb{R}^{n+1}$.

Let's recall the definition of parabolic BMO spaces. We say that $f \in L^1_{loc}$ is in the space $BMO(\mathbb{R}^{n+1})$ if the BMO seminorm

$$\| f \|_* = \sup_Q \frac{1}{|Q|} \int_Q |f(x') - f_Q| dx'$$

is finite, where Q ranges over the class of parabolic balls in $\mathbb{R}^{n+1}$ and $f_Q = \frac{1}{|Q|} \int_Q f(x') dx'$.

Let's recall some definitions and results of real analysis which hold in this "parabolic" context.

For $f \in L^1_{loc}(\mathbb{R}^{n+1})$, define the parabolic maximal function

$$Mf(x') = \sup_{Q \ni x'} \frac{1}{|Q|} \int_Q |f(y')| dy, \quad M_l f(x') = M(|f|^l)^{1/l}(x'),\ l > 0;$$

and the parabolic sharp function

$$M^\sharp f(x') = \sup_{Q \ni x'} \frac{1}{|Q|} \int_Q |f(y') - f_Q| dy',$$

where in both functions, the sup is taken over all parabolic balls Q in $\mathbb{R}^{n+1}$.

The next four well-known lemmas follow from results stated in [3].

Lemma 2.1 ([3], Maximal inequality). *For $f \in L^p$, $1 < p \leq \infty$, we have*

$$\| Mf \|_{L^p(\mathbb{R}^{n+1})} \leq C(p) \| f \|_{L^p(\mathbb{R}^{n+1})} .$$

Lemma 2.2 ([3], John-Nirenberg type lemma). *For $1 \leq p < \infty$, let $f \in BMO$ and Q be a parabolic ball, we have*

$$\left(\frac{1}{|Q|} \int_Q |f(y') - f_Q|^p dy' \right)^{\frac{1}{p}} \leq C(p) \| f \|_* .$$

Lemma 2.3 ([3], Sharp inequality). *For every p, $1 \leq p < \infty$, there exists a constant $C = C(p)$ such that if $f \in L^p$, then*

$$\| f \|_{L^p(\mathbb{R}^{n+1})} \leq C \| M^\sharp f \|_{L^p(\mathbb{R}^{n+1})} .$$

Lemma 2.4 ([3]). *Let $f \in BMO$. Then, for any positive integer j and parabolic ball Q*

$$\left|a_{2^j Q} - a_Q\right| \le C(n) j \parallel a \parallel_* .$$

Now we recall the definition and some properties of the auxiliary function $m_V(x)$; see [13] and [14] for $x \in \mathbb{R}^n$, the function $m_V(x)$ is defined by

$$\rho(x) \equiv \frac{1}{m_V(x)} = \sup_{r>0}\left\{r : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \le 1\right\}.$$

We have the following lemma about $m_V(x)$.

Lemma 2.5 (Lemma 1.4 in [14]). *Let $V \in B_q$ with $q \ge \frac{n}{2}$. For some positive integer k_0 and any $x, y \in \mathbb{R}^n$, we have:*

(a) $m_V(x) \sim m_V(y)$ *if* $|x-y| \le \frac{C}{m_V(x)}$,

(b) $m_V(y) \le C\left(1 + |x-y| m_V(x)\right)^{k_0} m_V(x)$,

(c) $m_V(y) \ge \frac{C m_V(x)}{(1+|x-y| m_V(x))^{k_0/(k_0+1)}}$.

In view of Corollary 1.5 in [14], the following inequalities hold

$$C\left\{1 + m_V(y)|x-y|\right\}^{\frac{1}{k_0}} \le 1 + m_V(y)|x-y| \le C\left\{1 + m_V(y)|x-y|\right\}^{k_0}.$$

Hence we can replace $m_V(x)$ with $m_V(y)$ possibly changing the integer k.

Using the Hölder inequality and the B_q condition we see that

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-2}} dy \le C \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy \tag{2.1}$$

and for $V \in B_n$,

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-1}} dy \le C \frac{1}{R^{n-1}} \int_{B(x,R)} V(y)dy. \tag{2.2}$$

Lemma 2.6 (Lemma 1.2 in [14]). *There exists a $C > 0$ such that, for any $0 < r < R < \infty$,*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \le C\left(\frac{R}{r}\right)^{\frac{n}{q}-2} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy.$$

Lemma 2.7 (Theorem 1 in [9]). *Let $V \in B_{n/2}$ and $\tau \in \mathbb{R}$. Suppose $\Gamma(x,t;y,s;\tau)$ is the fundamental solution to $\partial_t u - \Delta u + Vu = 0$ in $\mathbb{R}^{n+1}$. Then*

$$|\Gamma(x,t;y,s;\tau)| \le \frac{C_k}{(1 + m_V(x) d((x,t),(y,s)))^k} \frac{e^{-C_0|x-y|^2/(t-s)}}{(t-s)^{\frac{n}{2}}}$$

for any $(x,t),(y,s) \in \mathbb{R}^{n+1}$ and $t > s$, C_k is constant depending only on n, k and the constant in (1.5) and C_0 is constant depending only on n.

Lemma 2.8 (Lemma 2.3 in [2], pointwise Hörmander condition)**.** *Let K be a parabolic Calderón-Zygmund Kernel. Then for any parabolic ball Q of center x'_Q*

$$\left|K(x'-y') - K(x'_Q - y')\right| \leq C(K)\frac{d(x',x'_Q)}{d(x'_Q,y')^{n+3}}$$

for $x' \in Q, y' \notin 2Q$.

3. L^p estimates for small $a_{ij} \in BMO(\mathbb{R}^{n+1})$

Our main result in this section is as follows.

Theorem 3.1. *Suppose a_{ij} satisfying (1.1) and (1.2) for $i,j = 1,2,\ldots,n$, for every $1 < p \leq q$ and $V \in B_q$, there exists a constant $C > 0$ such that*

$$\| D_{ij}u \|_{L^p(\mathbb{R}^{n+1})} + \| Vu \|_{L^p(\mathbb{R}^{n+1})} \leq C \| Lu \|_{L^p(\mathbb{R}^{n+1})} \tag{3.1}$$

provided that $\| a_{ij} \|_* < \epsilon$ for small ϵ, where the constants C and ϵ depend on n, p, the ellipticity constant ς and the B_q constant of V.

We remark that A. Carbonaro, G. Metafune, C. Spina [4] proved L^p boundedness for Parabolic Schrödinger operators. Theorem 3.1 generalizes main results in [4] to discontinuous coefficients case. Of course, they in [4] considered more general V.

To prove Theorem 3.1, we need the following Theorem 3.2 proved essentially in [2]. Next, we will give another proof.

Theorem 3.2. *Suppose a_{ij} satisfying (1.1) and (1.2) for $i,j = 1,2,\ldots,n$. For any $1 < p < \infty$ there exists a positive constant C depending on n, p, the ellipticity constant ς, and the BMO seminorm of the leading coefficient such that*

$$\| D^2u \|_{L^p(\mathbb{R}^{n+1})} \leq C \| Au \|_{L^p(\mathbb{R}^{n+1})}$$

provided that $\| a_{ij} \|_* < \epsilon$ for small ϵ depending on n, p, the ellipticity constant ς and the B_q constant of V.

Proof. We first prove the following inequality

$$M^\sharp(D_{ij}u)(z) \leq C \| a_{ij} \|_*^{\frac{1}{lv'}} M_{lv}(D_{ij}u)(z) + CM_l(Au)(z) \tag{3.2}$$

for $1 < l < \infty$, $1 < v < \infty$ and $1/v' + 1/v = 1$.

For any parabolic ball Q in $\mathbb{R}^{n+1}$ containing the point $z \in \mathbb{R}^{n+1}$. Let $A_0u = \partial_t u - a_{ij}^Q u_{x_ix_j}$ with $a_{ij}^Q = \frac{1}{|Q|}\int_Q a_{ij}(y')dy'$. Let C be a constant to be fixed along the proof. We have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |D_{ij}u - C|dy' \\
= &\frac{1}{|Q|}\int_Q |D_{ij}A_0^{-1}(A_0u) - C|dy'
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|Q|}\int_Q |D_{ij}A_0^{-1}\left(A_0u\chi_{2Q} + A_0u\chi_{(2Q)^c}\right) - C|dy' \\
&\le \frac{1}{|Q|}\int_Q |D_{ij}A_0^{-1}(A_0u\chi_{2Q})|dy' + \frac{1}{|Q|}\int_Q |D_{ij}A_0^{-1}(A_0u\chi_{(2Q)^c}) - C|dy' \\
&:= I + II.
\end{aligned}
$$

Applying Hölder inequality and the $L^l(\mathbb{R}^{n+1})$ boundedness of $D_{ij}A_0^{-1}$ (since $D_{ij}A_0^{-1}$ is a parabolic Calderón-Zygmund operator), we obtain

$$
\begin{aligned}
I &\le \left(\frac{1}{|Q|}\int_Q |D_{ij}A_0^{-1}(A_0u\chi_{2Q})|^l dy'\right)^{\frac{1}{l}} \\
&\le C\left(\frac{1}{|Q|}\int_{2Q} |A_0u(y')|^l dy'\right)^{\frac{1}{l}} \\
&\le C\left(\frac{1}{|Q|}\int_{2Q} |A_0u(y') - Au(y')|^l dy'\right)^{\frac{1}{l}} + C\left(\frac{1}{|Q|}\int_{2Q} |Au(y'))|^l dy'\right)^{\frac{1}{l}} \\
&\le CI_1 + CM_l(Au)(z).
\end{aligned}
$$

Now let us estimate I_1.

$$
\begin{aligned}
I_1 &= \left(\frac{1}{|Q|}\int_{2Q} |A_0u - Au|^l dy'\right)^{\frac{1}{l}} \\
&= \left(\frac{1}{|Q|}\int_{2Q} |(a_{ij} - a_{ij}^Q)D_{ij}u|^l dy'\right)^{\frac{1}{l}} \\
&\le C\left(\frac{1}{|Q|}\int_{2Q} |a_{ij} - a_{ij}^Q|^{v'l} dy'\right)^{\frac{1}{v'l}} \cdot \left(\frac{1}{|Q|}\int_{2Q} |D_{ij}u|^{lv} dy'\right)^{\frac{1}{lv}} \\
&\le C \parallel a_{ij} \parallel_*^{\frac{1}{lv'}} M_{lv}(D_{ij}u)(z).
\end{aligned}
$$

Therefore we have

$$
I \le C \parallel a_{ij} \parallel_*^{\frac{1}{lv'}} M_{lv}(D_{ij}u)(z) + CM_l(Au)(z).
$$

Taking $C = D_{ij}A_0^{-1}(A_0u\chi_{(2Q)^c})(y_0')$, where y_0' is the center of the parabolic ball Q, we have

$$
\begin{aligned}
II &= \frac{1}{|Q|}\int_Q |D_{ij}A_0^{-1}(A_0u\chi_{(2Q)^c}) - C|dy' \\
&= \frac{1}{|Q|}\int_Q |D_{ij}A_0^{-1}(A_0u\chi_{(2Q)^c})(y') - D_{ij}A_0^{-1}(A_0u\chi_{(2Q)^c})(y_0')|dy'.
\end{aligned}
$$

Now we estimate $J = D_{ij}A_0^{-1}(A_0u\chi_{(2Q)^c})(y') - D_{ij}A_0^{-1}(A_0u\chi_{(2Q)^c})(y_0')$. Let K be the parabolic Calderón-Zygmund kernel of operator $D_{ij}A_0$. By Lemma 2.8, we have

$$
J \le \int_{2Q^c} |K(y' - z) - K(y_0' - z)||A_0u(z)|dz
$$

$$\begin{aligned}
&\leq \int_{2Q^c} \frac{d(y', y_0')}{d(y', z)^{n+3}} |A_0 u(z)| dz \\
&\leq \sum_{k=1}^{\infty} \int_{2^k r < \rho(z, y_0) < 2^{k+1} r} \frac{r}{(2^k r)^{n+3}} |A_0 u(z) - Au(z) + Au(z)| dz \\
&\leq C \sum_{k=1}^{\infty} 2^{-k} \left(\frac{1}{|2^{k+1} Q|} \int_{2^{k+1} Q} |A_0 u - Au|^l \, dz \right)^{\frac{1}{l}} + C M_l(Au)(z) \\
&\leq C \parallel a_{ij} \parallel_*^{\frac{1}{lv'}} M_{lv}(D_{ij} u)(z) + C M_l(Au)(z).
\end{aligned}$$

Thus (3.2) holds.

By Lemma 2.1, Lemma 2.3 and (3.2), if $lv < p$, we then get

$$\begin{aligned}
\parallel D_{ij} u \parallel_{L^p} &\leq \parallel M^{\sharp}(D_{ij} u) \parallel_{L^p} \\
&\leq C \parallel a_{ij} \parallel_*^{\frac{1}{lv'}} \parallel M_{lv}(D_{ij} u) \parallel_{L^p} + C \parallel M_l(Au) \parallel_{L^p} \\
&\leq C \parallel a_{ij} \parallel_*^{\frac{1}{lv'}} \parallel D_{ij} u \parallel_{L^p} + C \parallel Au \parallel_p .
\end{aligned}$$

Note that $\sum_{i,j=1}^n \parallel a_{ij} \parallel_*^{\frac{1}{lv'}} < n^2 (\epsilon)^{1/lv'}$ and taking $\varepsilon = (\frac{1}{2n^2 C})^{lv'}$, we can obtain the desired result. □

Theorem 3.3. *Under the assumptions* (1.1)-(1.3), *for any* $1 < p \leq q$ *there exists a positive constant* C *depending on* n, p, q, *the ellipticity constant* ς, *and the* B_q *constant of* V *such that*

$$\parallel Vu \parallel_{L^p(\mathbb{R}^{n+1})} \leq C \parallel Lu \parallel_{L^p(\mathbb{R}^{n+1})},$$

provided that $\|a_{ij}\|_*$ *is small.*

By Theorems 3.2 and 3.3, we have:

Corollary 3.1. *Suppose* a_{ij} *satisfying* (1.1)-(1.3) *for* $i, j = 1, 2, \ldots, n$. *For any* $1 < p \leq q$, *there exists a positive constant* C *depending on* n, p, *the ellipticity constant* ς *and the BMO seminorm of the leading coefficient such that*

$$\parallel D^2 u \parallel_{L^p(\mathbb{R}^{n+1})} \leq C \parallel Lu \parallel_{L^p(\mathbb{R}^{n+1})},$$

$\parallel a_{ij} \parallel_* < \epsilon$ *for small* ϵ.

To prove Theorem 3.3, we freeze the coefficients of A at x_0 and get the operator

$$L_0 u(x') = u_t(x') - a_{ij}(x_0, t_0) u_{x_i x_j}(x') + V(x) u(x').$$

Let $A_0 u(x') = u_t(x') - a_{ij}(x_0, t_0) u_{x_i x_j}(x')$. For any $u \in C_0^\infty(\mathbb{R}^{n+1})$, $x' \in \mathbb{R}^{n+1}$, we can write:

$$\begin{aligned}
u(x, t) &= \int_{-\infty}^t \int_{\mathbb{R}^n} \Gamma(x, t; y, s) L_0 u(y, s) dy ds \\
&= \int_{-\infty}^t \int_{\mathbb{R}^n} \Gamma(x, t; y, s) Lu(y, s) dy ds
\end{aligned}$$

$$+\int_{-\infty}^{t}\int_{\mathbb{R}^n}\Gamma(x,t;y,s)[A_0u(y,s)-Au(y,s)]dyds.$$

Let $x_0=x$, we have:

$$\begin{aligned}u(x,t) = & \int_{-\infty}^{t}\int_{\mathbb{R}^n}\Gamma(x,t;y,s)Lu(y,s)dyds \\ & + \sum_{i,j=1}^{n}\int_{-\infty}^{t}\int_{\mathbb{R}^n}\Gamma(x,t;y,s)[a_{ij}(y,s)-a_{ij}(x,t)]u_{y_iy_j}(y,s)dyds.\end{aligned}$$

For every positive integer k, by Lemma 2.7 we get:

$$\begin{aligned}| V(x)u(x,t) | \leq & C_kV(x)\int_{-\infty}^{t}\int_{\mathbb{R}^n}\frac{1}{(1+m_V(x)d((x,t),(y,s)))^k}\frac{e^{-C_0|x-y|^2/(t-s)}}{(t-s)^{\frac{n}{2}}} \\ & \times\left(|Lu(y,s)|+\sum_{i,j=1}^{n}|a_{ij}(y,s)-a_{ij}(x,t)||u_{y_iy_j}(y,s)|\right)dyds \\ \leq & C_kV(x)\int_{-\infty}^{t}\int_{\mathbb{R}^n}\frac{1}{(1+m_V(x)d((x,t),(y,s)))^k}\cdot\frac{1}{d((x,t),(y,s))^n} \\ & \times\left(|Lu(y,s)|+\sum_{i,j=1}^{n}|a_{ij}(y,s)-a_{ij}(x,t)|u_{y_iy_j}(y,s)|\right)dyds.\end{aligned}$$

For any $x'=(x,t)\in\mathbb{R}^{n+1}$ and any positive integer k. Let us introduce the integral operators:

$$S_kf(x')=V(x)\int_{-\infty}^{t}\int_{\mathbb{R}^n}\frac{1}{(1+m_V(x)d((x,t),(y,s)))^k}\cdot\frac{1}{d((x,t),(y,s))^n}f(y,s)dyds;$$

$$\begin{aligned}S_{k,a}f(x')=V(x)\int_{-\infty}^{t}\int_{\mathbb{R}^n} & \frac{1}{(1+m_V(x)d((x,t),(y,s)))^k}\cdot\frac{1}{d((x,t),(y,s))^n}f(y,s) \\ & |a(y,s)-a(x,t)|\,dyds.\end{aligned}$$

Then, we have

$$| V(x)u(x,t) | \leq C_kS_k(|Lu|)(x,t)+\sum_{i,j=1}^{n}S_{k,a_{ij}}(|u_{x_ix_j}|)(x,t). \tag{3.3}$$

We will prove that for any $1<p\leq q$ and for k large enough

$$\| S_kf \|_{L^p(\mathbb{R}^{n+1})}\leq C \| f \|_{L^p(\mathbb{R}^{n+1})} \tag{3.4}$$

and

$$\| S_{k,a}f \|_{L^p(\mathbb{R}^{n+1})}\leq C\|a\|_* \| f \|_{L^p(\mathbb{R}^{n+1})} . \tag{3.5}$$

Now, by (3.3)-(3.5) and Theorem 3.2, for any $u\in C_0^\infty(\mathbb{R}^{n+1})$, we have

$$\| Vu \|_{L^p} \leq C \| Lu \|_{L^p} +C\epsilon\sum_{i,j=1}^{n} \| u_{x_ix_j} \|_{L^p}$$

$$\leq C \parallel Lu \parallel_{L^p} + Cn^2\epsilon \parallel Au \parallel_{L^p}$$
$$\leq (C + Cn^2\epsilon) \parallel Lu \parallel_{L^p} + Cn^2\epsilon \parallel Vu \parallel_{L^p} .$$

If $\epsilon \leq \frac{1}{2n^2C}$, we get Theorem 3.3.

In order to do that, it is more convenient to consider the transposed operator:

$$S_k^* f(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} \frac{V(y)}{(1 + m_V(x) d((x,t),(y,s)))^k} \cdot \frac{1}{d((x,t),(y,s))^n} f(y,s) dy ds;$$
$$S_{k,a}^* f(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} \frac{V(y)}{(1 + m_V(x) d((x,t),(y,s)))^k} \cdot \frac{1}{d((x,t),(y,s))^n} f(y,s) \, |a(y,s) - a(x,t)| \, dy ds.$$

Proposition 3.1. *Let $V \in B_q$, $q \geq \frac{n}{2}$. For k large enough and $q' \leq p < \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$, the operator S_k^* is continuous on $L^p(\mathbb{R}^{n+1})$.*

Proposition 3.2. *Suppose $V \in B_q$, $q \geq \frac{n}{2}$ and $a \in BMO$. For k large enough and $q' \leq p < \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$, then there exists a constant C such that*

$$\parallel S_{k,a}^* f \parallel_{L^p(\mathbb{R}^{n+1})} \leq C \|a\|_* \parallel f \parallel_{L^p(\mathbb{R}^{n+1})} .$$

By duality, the above propositions imply (3.4), (3.5). Therefore, the rest of this section will be devoted to the proof of Proposition 3.1 and Proposition 3.2.

Proof of Proposition 3.1. Since the kernel is positive, we can also assume $f \geq 0$. Also, we may assume $q > n/2$ because of the property $B_q \Rightarrow B_{q+\epsilon}$ for some $\epsilon > 0$.

We will prove the following pointwise bound

$$S_k^* f(x,t) \leq C M_{q'} f(x,t), \tag{3.6}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

By the maximal inequality, for $p > q'$, (3.6) implies Proposition 3.1. When $p = q'$, we apply again the fact that actually $V \in B_{q+\epsilon}$ for some $\epsilon > 0$, as already noted, so that (3.6) also holds with a smaller q'.

$$\begin{aligned} S_k^* f(x,t) &\leq C \int_{d((x,t),(y,s))<\rho(x)} \frac{V(y)}{\left(1 + \frac{d((x,t),(y,s))}{\rho(x)}\right)^k} \cdot \frac{1}{d((x,t),(y,s))^n} f(y,s) dy ds \\ &\quad + C \int_{d((x,t),(y,s))\geq\rho(x)} \frac{V(y)}{\left(1 + \frac{d((x,t),(y,s))}{\rho(x)}\right)^k} \cdot \frac{1}{d((x,t),(y,s))^n} f(y,s) dy ds \\ &\leq C \int_{d((x,t),(y,s))<\rho(x)} \frac{V(y)}{d((x,t),(y,s))^n} f(y,s) dy ds \\ &\quad + C \int_{d((x,t),(y,s))\geq\rho(x)} \left(\frac{\rho(x)}{d((x,t),(y,s))}\right)^k \frac{V(y)}{d((x,t),(y,s))^n} f(y,s) dy ds \\ &\equiv A(x,t) + B(x,t). \end{aligned}$$

Let $Q_j = Q((x,t), 2^{-j}\rho(x))$. By Hölder inequality and $V \in B_q$, we obtain

$$\begin{aligned}
A(x,t) &\leq C \sum_{j=0}^{\infty} \frac{1}{(2^{-j}\rho(x))^n} \int_{d((x,t),(y,s))\simeq 2^{-j}\rho(x)} V(y) f(y,s)dyds \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{(2^{-j}\rho(x))^n} \left(\int_{Q_j} V(y)^q dyds \right)^{\frac{1}{q}} \left(\int_{Q_j} |f(y,s)|^{q'} dyds \right)^{\frac{1}{q'}} \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{(2^{-j}\rho(x))^n} \left(2^{-j}\rho(x)\right)^{(n+2)/q'} \left(2^{-j}\rho(x)\right)^{(n+2)/q} \\
&\quad \left(\frac{1}{|B(x,2^{-j}\rho(x))|} \int_{B(x,2^{-j}\rho(x))} V(y)dyds \right) \\
&\quad \times \left(\frac{1}{(2^{-j}\rho(x))^{(n+2)}} \int_{Q_j} |f(y,s)|^{q'} dyds \right)^{\frac{1}{q'}} \\
&\leq CM_{q'}f(x,t) \sum_{j=0}^{\infty} \left(2^{-j}\rho(x)\right)^2 \left(\frac{1}{|B(x,2^{-j}\rho(x))|} \int_{B(x,2^{-j}\rho(x))} V(y)dyds \right),
\end{aligned}$$

here and in what follows, $d((x,t),(y,s)) \approx 2^j r$ denotes $2^j r \leq d((x,t),(y,s)) \leq 2^{j+1} r$. By Lemma 2.6, we have

$$\frac{1}{r^n} \int_{B(x,r)} V(y)dy \leq C \left(\frac{R}{r} \right)^{\frac{n}{q}} \frac{1}{R^n} \int_{B(x,r)} V(y)dy \tag{3.7}$$

for any $0 < r < R < \infty$. Taking $R = \rho(x)$ and $r = 2^{-j}\rho(x)$ in (3.5), we obtain

$$\begin{aligned}
A(x,t) &\leq CM_{q'}f(x,t) \sum_{j=0}^{\infty} \left(2^{-j}\rho(x)\right)^2 \left(2^j\right)^{\frac{n}{q}} \left(\frac{1}{|B(x,\rho(x))|} \int_{B(x,\rho(x))} V(y)dy \right) \\
&\leq CM_{q'}f(x,t) \left(\frac{1}{\rho(x)^{n-2}} \int_{B(x,\rho(x))} V(y)dy \right) \sum_{j=0}^{\infty} \left(2^{-j}\right)^{2-\frac{n}{q}} \\
&\leq CM_{q'}f(x,t),
\end{aligned}$$

where up to the last inequality applies $\frac{1}{\rho(x)^{n-2}} \int_{B(x,\rho(x))} V(y)dy \leq 1$.

Similar to the estimates of $A(x,t)$, let $Q_j = Q((x,t), 2^j\rho(x))$ and we obtain

$$\begin{aligned}
B(x,t) &\leq C \sum_{j=0}^{\infty} \frac{2^{-jk}}{(2^j\rho(x))^n} \int_{d((x,t),(y,s))\simeq 2^j\rho(x)} V(y) f(y,s)dyds \\
&\leq C \sum_{j=0}^{\infty} \frac{2^{-jk}}{(2^j\rho(x))^n} \left(\int_{Q_j} V(y)^q dyds \right)^{\frac{1}{q}} \left(\int_{Q_j} |f(y,s)|^{q'} dyds \right)^{\frac{1}{q'}}
\end{aligned}$$

$$\leq CM_{q'}f(x,t)\sum_{j=0}^{\infty}\frac{(2^j\rho(x))^2}{2^{jk}}\left(\frac{1}{|B(x,2^j\rho(x))|}\int_{B(x,2^j\rho(x))}V(y)dyds\right).$$

Since $V(y)dy$ is doubling, for some positive constant α, C and all j, we have

$$\int_{B(x,2^j\rho(x))}V(y)dy\leq C2^{\alpha j}\int_{B(x,\rho(x))}V(y)dy.$$

Thus

$$\begin{aligned}B(x,t)&\leq CM_{q'}f(x,t)\sum_{j=0}^{\infty}\frac{\left(2^j\rho(x)\right)^2}{2^{jk}}\frac{2^{\alpha j}}{\left(2^j\rho(x)\right)^n}\int_{B(x,\rho(x))}V(y)dy\\&=CM_{q'}f(x,t)\frac{1}{\rho(x)^{n-2}}\int_{B(x,\rho(x))}V(y)dy\sum_{j=0}^{\infty}\frac{1}{2^{j(k+n-\alpha-2)}}\\&\leq CM_{q'}f(x,t),\end{aligned}$$

where we used again $\frac{1}{\rho(x)^{n-2}}\int_{B(x,\rho(x))}V(y)dy\leq 1$ and we have chosen k large enough to get $(k+n-\alpha-2)$ positive. The proof is complete. □

In order to prove Proposition 3.2, it is convenient to settle this result in a suitably abstract framework. Let

$$\omega((x,t),(y,s))=\frac{V(y)}{\left(1+\frac{d((x,t),(y,s))}{\rho(y)}\right)}\cdot\frac{1}{d((x,t),(y,s))^n}\tag{3.8}$$

be the kernel of the integral operator S_k^*, so that

$$S_{k,a}^*f(x,t)=\int\omega((x,t),(y,s))|a(y,s)-a(x,t)|f(y,s)dyds.$$

We will deduce Proposition 3.2 from an abstract result.

Definition 3.1. We say that the kernel $W((x,t),(y,s))$ satisfies "Hörmander's condition of order q" in the first variable, briefly $W\in H_1(q)$ if there exists a constant C such that for any $r>0$

$$\sum_{j=1}^{\infty}j\left(2^jr\right)^{(n+2)/q'}\left(\int_{2^jr\leq d((y,s),(x_0,t_0))\leq 2^{j+1}r}|W((x,t),(y,s))-W((x_0,t_0),(y,s))|^q\,dyds\right)^{\frac{1}{q}}\leq C.$$

Proposition 3.3. *Let $W((x,t),(y,s))$ be a nonnegative kernel satisfying $H_1(q)$ for some $q>1$ and such that the integral operator*

$$Tf(x,t)=\int_{\mathbb{R}^{n+1}}W((x,t),(y,s))f((y,s))dyds$$

is continuous on $L^p(\mathbb{R}^{n+1})$ for any $q'<p<\infty$. Then for $b\in BMO(\mathbb{R}^{n+1})$ the operator

$$T_bf(x,t)=\int_{\mathbb{R}^{n+1}}|b(x,t)-b(y,s)|\,W((x,t),(y,s))f(y,s)dyds$$

is bounded on $L^p(\mathbb{R}^{n+1})$ *for any* $q' < p < \infty$ *and*

$$\| T_b f \|_p \leq C \| b \|_* \| f \|_p,$$

where $\| b \|_*$ *stands for the BMO seminorm.*

Proof of Proposition 3.3. We may assume $b \in L^\infty(\mathbb{R}^{n+1})$ to prove this proposition and then remove it by a standard truncation.

We will prove the following pointwise inequality: for any $h > q'$, there exists a constant C such that

$$M^\sharp(T_b f)(z) \leq C \| b \|_* \left[M_h(Tf)(z) + (M_h f)(z) \right] \tag{3.9}$$

with C independent of b and f.

Let $Q = Q((x_0, t_0), r) = \{(y,s) \in \mathbb{R}^{n+1} : |y - x_0| < r, |s - t_0| < r^2\}$ be a parabolic ball such that $z \in Q$. Let $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$, $f \geq 0$. For any $(x,t) \in Q$

$$|T_b f(x,t) - C| \leq |T_b f_1(x,t)| + |T_b f_2(x,t) - C| = I + II.$$

For the first term, we obtain

$$\begin{aligned} I &= \left| \int_{-\infty}^{t} \int_{\mathbb{R}^n} |b(x,t) - b(y,s)| \, W((x,t),(y,s)) f_1(y,s) dy ds \right| \\ &\leq |b(x,t) - b_Q| T f_1(x,t) + T\left(|b - b_Q| f_1(x,t) \right). \end{aligned}$$

Choosing $C = \int |b(y,s) - b_Q| \, W((x_0,t_0),(y,s)) dy ds$. Then,

$$\begin{aligned} II &= |T_b f_2(x,t) - C| \\ &= \left| T_b f_2(x,t) - \int_{-\infty}^{t} \int_{\mathbb{R}^n} |b(y,s) - b_Q| \, W((x_0,t_0),(y,s)) f_2(y,s) dy ds \right| \\ &\leq \int_{-\infty}^{t} \int_{\mathbb{R}^n} \big| |b(x,t) - b(y,s)| W((x,t),(y,s)) \\ &\qquad - |b(y,s) - b_Q| W((x_0,t_0),(y,s)) \big| \, f_2(y,s) dy ds \\ &\leq \int_{-\infty}^{t} \int_{\mathbb{R}^n} \big| |b(x,t) - b(y,s)| - |b(y,s) - b_Q| \big| \, W((x,t),(y,s)) f_2(y,s) dy ds \\ &\qquad + \int_{-\infty}^{t} \int_{\mathbb{R}^n} |b(y,s) - b_Q| \, |W((x,t),(y,s)) \\ &\qquad - W((x_0,t_0),(y,s))| \, f_2(y,s) dy ds \\ &\leq \int_{-\infty}^{t} \int_{\mathbb{R}^n} |b(x,t) - b_Q| \, W((x,t),(y,s)) f_2(y,s) dy ds \\ &\quad + \int_{-\infty}^{t} \int_{\mathbb{R}^n} |b(y,s) - b_Q| \, |W((x,t),(y,s)) - W((x_0,t_0),(y,s))| \, f_2(y,s) dy ds. \end{aligned}$$

Thus we have

$$|T_b f(x,t) - C|$$

$$\begin{aligned}&\le |b(x,t)-b_Q|Tf(x,t)+T(|b-b_Q|f_1)(x,t)\\&\quad+\int |b(y,s)-b_Q|\,|W((x,t),(y,s))-W((x_0,t_0),(y,s))|\,f_2(y,s)dyds\\&\equiv A(x,t)+B(x,t)+C(x,t).\end{aligned}$$

For the first term, by Lemma 2.2, we get

$$\begin{aligned}&\frac{1}{|Q|}\int_Q A(x,t)dxdt\\&=\frac{1}{|Q|}\int_Q |b(x,t)-b_Q|Tf(x,t)dxdt\\&\le \left(\frac{1}{|Q|}\int_Q |b(x,t)-b_Q|^{h'}dxdt\right)^{\frac{1}{h'}}\left(\frac{1}{|Q|}\int_Q |Tf(x,t)|^h dxdt\right)^{\frac{1}{h}}\\&\le \parallel b\parallel_* M_h(Tf)(z).\end{aligned}$$

Next, we choose γ such that $h>\gamma>q'$. Then

$$\begin{aligned}&\frac{1}{|Q|}\int_Q B(x,t)dxdt\\&=\frac{1}{|Q|}\int_Q T(|b-b_Q|f_1)(x,t)dxdt\\&\le \left(\frac{1}{|Q|}\int_Q T(|b-b_Q|f_1)^\gamma(x,t)dxdt\right)^{\frac{1}{\gamma}}\\&\le C\left(\frac{1}{|Q|}\int_{2Q}|b-b_Q|^\gamma|f_1(x,t)|^\gamma dxdt\right)^{\frac{1}{\gamma}}\\&\le C\left(\frac{1}{|Q|}\int_{2Q}|f(x,t)|^h dxdt\right)^{\frac{1}{h}}\left(\frac{1}{|Q|}\int_{2Q}|b-b_Q|^{\gamma(\frac{h}{\gamma}')}dxdt\right)^{\frac{1}{\gamma(\frac{h}{\gamma})'}}\\&\le C\biggl(\frac{1}{|2Q|}\int_{2Q}|f(x,t)|^h dxdt\biggr)^{\frac{1}{h}}\left\{\left(\frac{1}{|2Q|}\int_{2Q}|b-b_Q|^{\gamma(\frac{h}{\gamma})'}dxdt\right)^{\frac{1}{\gamma(\frac{h}{\gamma})'}}+|b_Q-b_{2Q}|\right\}\\&\le C\parallel b\parallel_* M_h(f)(z),\end{aligned}$$

where in the last inequality we have used $|b_Q-b_{2Q}|\le C\parallel b\parallel_*$ and $\frac{1}{(h/\gamma)'}+\frac{1}{(h/\gamma)}=1$. Finally, since $h>q'$, we choose γ such that $\frac{1}{\gamma}+\frac{1}{q}+\frac{1}{h}=1$. Let $Q_j=Q((x_0,t_0),2^jr)$. For any $(x,t)\in Q$, applying Hölder inequality, Lemma 2.4 and the $H_1(q)$ condition on $W((x,t),(y,s))$, we have

$$\begin{aligned}&C(x,t)\\&=\int_{d((x_0,t_0),(y,s))\ge 2r}|b(y,s)-b_Q||W((x,t),(y,s))-W((x_0,t_0),(y,s))|f(y,s)dyds\\&=\sum_{j=2}^{\infty}\int_{d((x_0,t_0),(y,s))\approx 2^jr}|b(y,s)-b_Q||W((x,t),(y,s))-W((x_0,t_0),(y,s))|f(y,s)dyds\end{aligned}$$

$$\leq C\sum_{j=2}^{\infty}\left(\int_{d((x,t),(y,s))\approx 2^j r}|b(y,s)-b_Q|^{\gamma}dyds\right)^{\frac{1}{\gamma}}\left(\int_{d((x_0,t_0),(y,s))\approx 2^j r}|f(y,s)|^{h}dyds\right)^{\frac{1}{h}}$$

$$\times\left(\int_{d((x_0,t_0),(y,s))\approx 2^j r}|W((x,t)(y,s))-W(x_0,t_0),(y,s))|^{q}dyds\right)^{\frac{1}{q}}$$

$$\leq C\sum_{j=2}^{\infty}(2^j r)^{n+2}\left\{\left(\frac{1}{|Q_j|}\int_{d((x_0,t_0),(y,s))\approx 2^j r}|b-b_{Q_j}|^{\gamma}dyds\right)^{\frac{1}{\gamma}}+|b_Q-b_{Q_j}|\right\}$$

$$\times\left(\frac{1}{|Q_j|}\int_{d((x_0,t_0),(y,s))\approx 2^j r}|W((x,t),(y,s))-W((x_0,t_0),(y,s))|^{q}dyds\right)^{\frac{1}{q}}M_h f(z)$$

$$\leq C\parallel b\parallel_* M_h f(z)\sum_{j=2}^{\infty}(2^j r)^{(n+2)/q'}j$$

$$\times\left(\int_{d((x_0,t_0),(y,s))\approx 2^j r}|W((x,t),(y,s))-W((x_0,t_0),(y,s)|^{q}dyds\right)^{\frac{1}{q}}$$

$$\leq C\parallel b\parallel_* M_h f(z).$$

□

Proposition 3.4. *The kernel* $\omega((x,t),(y,s))$ *in* (3.8) *satisfies condition* $H_1(q)$.

Proof. Because of the property $B_q \Rightarrow B_{q+\epsilon}$ for some $\epsilon > 0$, we may assume $q > \frac{n}{2}$. Let (x,t), (y,s), (x_0,t_0) be such that $d((x,t),(x_0,t_0)) \leq r$ and $d((y,s),(x_0,t_0)) \geq 2r$. Thus $d((y,s),(x_0,t_0)) \approx d((y,s),(x,t))$. Then

$$|W((x,t),(y,s))-W((x_0,t_0),(y,s))|$$

$$\leq C_k V(y)\left[\frac{1}{(1+m_V(y)d((x_0,t_0),(y,s)))^k}\times\left|\frac{1}{d((x,t),(y,s))^n}-\frac{1}{d((x_0,t_0),(y,s))^n}\right|\right.$$

$$\left.+\frac{1}{d((x,t),(y,s))^n}\left|\frac{1}{(1+m_V(y)d((x,t),(y,s)))^k}-\frac{1}{(1+m_V(y)d((x_0,t_0),(y,s)))^k}\right|\right]$$

$$\equiv A+B.$$

For the term A, we have

$$A\leq\frac{C_k V(y)}{(1+m_V(y)d((x_0,t_0),(y,s)))^k}\times\frac{d((x,t),(x_0,t_0))}{d((x_0,t_0),(y,s))^{n+1}}.$$

For the term B, we have

$$B\leq\frac{C_k V(y)}{d((x,t),(y,s))^n}\times\frac{|d((x,t),(y,s))-d((x_0,t_0),(y,s))|m_V(y)}{(1+m_V(y)d((x_0,t_0),(y,s)))^{k+1}}.$$

In the estimates of B, we have used the following inequality:

$$\left|\frac{1}{(1+bt)^k}-\frac{1}{(1+bt_0)^k}\right|\leq\frac{kb}{(1+b\bar{t})^{k+1}}|t-t_0|$$

for some $\bar{t} \in [t_0, t]$. Now,

$$\left(\int_{2^j r \le d((x_0,t_0),(y,s)) \le 2^{j+1} r} |\omega((x,t),(y,s)) - \omega((x_0,t_0),(y,s))|^q \, dyds \right)^{\frac{1}{q}}$$
$$\le \left(\int_{2^j r \le d((x_0,t_0),(y,s)) \le 2^{j+1} r} A^q dyds \right)^{\frac{1}{q}} + \left(\int_{2^j r \le d((x_0,t_0),(y,s)) \le 2^{j+1} r} B^q dyds \right)^{\frac{1}{q}}.$$

We obtain

$$\left(\int_{2^j r \le d((x_0,t_0),(y,s)) \le 2^{j+1} r} A^q dyds \right)^{\frac{1}{q}}$$
$$\le \frac{C_k r}{(1+m_V(x)2^j r)^k (2^j r)^{n+1}} \left(\int_{2^j r \le d((x_0,t_0),(y,s)) \le 2^{j+1} r} V(y)^q dyds \right)^{\frac{1}{q}}$$
$$\le \frac{C_k}{(1+m_V(x)2^j r)^k} \times \frac{r}{(2^j r)^{n+1}} (2^j r)^{(n+2)/q-n} \int_{B(x_0, 2^{j+1} r)} V(y) dy,$$

so we have

$$\sum_j j(2^j r)^{(n+2)/q'} \left(\int_{2^j r \le d((x_0,t_0),(y,s)) \le 2^{j+1} r} A^q dyds \right)^{\frac{1}{q}}$$
$$\le \sum_j j(2^j r)^{(n+2)/q'} \times \frac{C_k}{(1+m_V(x)2^j r)^k}$$
$$\times \frac{r}{(2^j r)^{n+1}} (2^j r)^{(n+2)/q-n} \int_{B(x_0, 2^{j+1} r)} V(y) dy$$
$$= C_k \sum_j \frac{j}{(1+m_V(x)2^j r)^k} \times \frac{r}{(2^j r)^{n-1}} \int_{B(x_0, 2^{j+1} r)} V(y) dy$$
$$= C_k \sum_{j: 2^j r < \rho(x)} \frac{j}{(1+m_V(x)2^j r)^k} \times \frac{r}{(2^j r)^{n-1}} \int_{B(x_0, 2^{j+1} r)} V(y) dy$$
$$+ C_k \sum_{j: 2^j r \ge \rho(x)} \frac{j}{(1+m_V(x)2^j r)^k} \times \frac{r}{(2^j r)^{n-1}} \int_{B(x_0, 2^{j+1} r)} V(y) dy$$
$$\equiv A_1 + A_2.$$

By (3.5) and the definition of $\rho(x)$, we have

$$A_1 \le C_k \sum_{j: 2^j r < \rho(x)} \frac{j}{2^j} \frac{1}{(2^j r)^{n-2}} \int_{|x_0 - y| \le 2^{j+1} r} V(y) dy$$
$$\le C_k \sum_{j: 2^j r < \rho(x)} \frac{j}{2^j} \left(\frac{\rho(x)}{2^j r} \right)^{\frac{n}{q}-2} \frac{1}{\rho(x)^{n-2}} \int_{|x_0 - y| \le \rho(x)} V(y) dy$$

$$\le C_k \sum_{j:2^j r<\rho(x)} \frac{j}{2^j}\left(\frac{\rho(x)}{2^j r}\right)^{\frac{n}{q}-2}$$
$$\le C_k \sum_{j:2^j r<\rho(x)} \frac{j}{2^j} \le C_k.$$

By the doubling condition on $V(y)dy$, the definition of $\rho(x)$ and taking k large enough, we obtain

$$\begin{aligned}
A_2 &\le C_k \sum_{j:2^j r\ge\rho(x)} \frac{j}{2^j}\left(\frac{\rho(x)}{2^j r}\right)^{k} \frac{1}{(2^j r)^{n-2}} \int_{|x_0-y|\le 2^{j+1}r} V(y)dy \\
&\le C_k \sum_{j:2^j r\ge\rho(x)} \frac{j}{2^j}\left(\frac{\rho(x)}{2^j r}\right)^{k} \frac{1}{(2^j r)^{n-2}} \left(\frac{2^j r}{\rho(x)}\right)^{\alpha} \int_{|x_0-y|\le \rho(x)} V(y)dy \\
&\le C_k \sum_{j:2^j r\ge\rho(x)} \frac{j}{2^j}\left(\frac{\rho(x)}{2^j r}\right)^{k} \frac{1}{(2^j r)^{n-2}} \left(\frac{2^j r}{\rho(x)}\right)^{\alpha} \rho(x)^{n-2} \\
&= C_k \sum_{j:2^j r\ge\rho(x)} \frac{j}{2^j}\left(\frac{\rho(x)}{2^j r}\right)^{k-(n-2-\alpha)} \\
&\le C_k \sum_{j:2^j r\ge\rho(x)} \frac{j}{2^j} \le C_k.
\end{aligned}$$

We now check the analogous condition on the term B.

$$\begin{aligned}
&\left(\int_{2^j r\le d((x_0,t_0),(y,s))\le 2^{j+1}r} B^q dyds\right)^{\frac{1}{q}} \\
&\le C_k \frac{r m_V(x)}{(2^j r)^n} \frac{1}{(1+m_V(x)2^j r)^{k+1}} \left(\int_{2^j r\le d((x_0,t_0),(y,s))\le 2^{j+1}r} V(y)^q dyds\right)^{\frac{1}{q}} \\
&= \frac{C_k r m_V(x)}{(1+m_V(x)2^j r)^{k+1}} (2^j r)^{(n+2)/q-2n} \int_{B(x_0,2^{j+1}r)} V(y)dy.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\sum_j j(2^j r)^{(n+2)/q'} \left(\int_{2^j r\le d((x_0,t_0),(y,s))\le 2^{j+1}r} B^q dyds\right)^{\frac{1}{q}} \\
&\le \sum_j j(2^j r)^{(n+2)/q'} \frac{C_k r m_V(x)}{(1+m_V(x)2^j r)^{k+1}} (2^j r)^{(n+2)/q-2n} \int_{B(x_0,2^{j+1}r)} V(y)dy \\
&= C_k \sum_j \frac{j}{(2^j r)^{n-2}} \times \frac{r}{\rho(x)} \times \frac{1}{(1+\frac{2^j r}{\rho(x)})^{k+1}} \int_{B(x_0,2^{j+1}r)} V(y)dy
\end{aligned}$$

$$\begin{aligned}
&= C_k \sum_{j:2^j r<\rho(x)} \frac{j}{(2^j r)^{n-2}} \times \frac{r}{\rho(x)} \times \frac{1}{(1+\frac{2^j r}{\rho(x)})^{k+1}} \int_{B(x_0,2^{j+1}r)} V(y)dy \\
&\quad + C_k \sum_{j:2^j r\geq\rho(x)} \frac{j}{(2^j r)^{n-2}} \times \frac{r}{\rho(x)} \times \frac{1}{(1+\frac{2^j r}{\rho(x)})^{k+1}} \int_{B(x_0,2^{j+1}r)} V(y)dy \\
&\equiv B_1 + B_2.
\end{aligned}$$

For the term B_1, we have

$$\begin{aligned}
B_1 &\leq C_k \sum_{j:2^j r<\rho(x)} \frac{j}{(2^j r)^{n+2}} \times \frac{r}{\rho(x)} \times \int_{|x_0-y|\leq 2^{j+1}r} V(y)dy \\
&\leq C_k \sum_{j:2^j r<\rho(x)} \frac{j}{2^j} \times \frac{1}{(2^j r)^{n-2}} \int_{|x_0-y|\leq 2^{j+1}r} V(y)dy,
\end{aligned}$$

and from now on the estimate is the same as for the term A_1.

$$\begin{aligned}
B_2 &\leq \sum_{j:2^j r\geq\rho(x)} j\frac{C_k}{(2^j r)^{n-2}} \times \frac{r}{\rho(x)} \times \left(\frac{\rho(x)}{2^j r}\right)^{k+1} \int_{|x_0-y|\leq 2^{j+1}r} V(y)dy \\
&\leq \sum_{j:2^j r\geq\rho(x)} \frac{j}{2^j} \times \frac{C_k}{(2^j r)^{n-2}} \times \left(\frac{\rho(x)}{2^j r}\right)^{k} \int_{|x_0-y|\leq 2^{j+1}r} V(y)dy,
\end{aligned}$$

and from now on the estimate is the same as for the term A_2. □

Proof of Proposition 3.2. By Proposition 3.3 and Proposition 3.4 we get that for k large enough and $q' < p < \infty$, we have

$$\| S^*_{k,a} f \|_{L^p(\mathbb{R}^{n+1})} \leq C \| a \|_* \| f \|_{L^p(\mathbb{R}^{n+1})}$$

when $p = q'$ we still use the fact that actually $V \in B_{q+\epsilon}$. □

Now we turn to prove Theorem 3.1. In fact, by Theorem 3.3 and Corollary 3.1, we can get (3.1). □

4. Global L^p estimates for $a_{ij} \in VMO(\mathbb{R}^{n+1})$

In this section, we consider the global L^p estimates when $a_{ij} \in VMO(\mathbb{R}^{n+1})$. We first give two lemmas.

Lemma 4.1. *Under the assumptions* (1.1), (1.3) *and* (1.6) *for any* $1 < p \leq q$, *there exist constants* C *and* r *such that for any* $z_0 \in \mathbb{R}^{n+1}$, $u \in C_0^\infty(B_r(z_0))$

$$\| Vu \|_{L^p(B_r(z_0))} \leq C \| Lu \|_{L^p(B_r(z_0))} .$$

The constants C *and* r *depend on* n, p, q, *the ellipticity constant* ς, *the VMO moduli of the leading coefficients and the* B_q *constant of* V.

Since the proof of Lemma 4.1 is very similar to the proof of Theorem 3.3, we omit the details here.

M. Bramanti, M. C. Cerutti in [2] proved the following result.

Lemma 4.2. *Under the assumptions* (1.1) *and* (1.6)*, for any* $1 < p < \infty$*, there exist constants* C *and* r *such that for any* $z_0 \in \mathbb{R}^{n+1}$*,* $u \in C_0^\infty(B_r(z_0))$*,*

$$\| D^2 u \|_{L^p(B_r(z_0))} \leq C \| Au \|_{L^p(B_r(z_0))} .$$

The constants C *and* r *depend on* n*,* p*,* q*, the ellipticity constant* ς*, the VMO moduli of the leading coefficients.*

Now we state the first main result in this section.

Theorem 4.1. *Under the assumptions* (1.1)*,* (1.3) *and* (1.6)*, for every* $1 < p \leq q$*, there exists a constant* $C > 0$ *such that*

$$\| u \|_{W_p^{1,2}(\mathbb{R}^{n+1})} + \| Vu \|_{L^p(\mathbb{R}^{n+1})} \leq C \left\{ \| Lu \|_{L^p(\mathbb{R}^{n+1})} + \| u \|_{L^p(\mathbb{R}^{n+1})} \right\} \tag{4.1}$$

for any $u \in C_0^\infty(\mathbb{R}^{n+1})$*. The constant* C *depends on* n, p, q*, the ellipticity constant* ς*, the VMO moduli of the leading coefficients, and the* B_q *constant of* V*.*

The bound (4.1) immediately extends to all functions $u \in W_{p,V}^{1,2}(\mathbb{R}^{n+1})$, we define that the closure of $C_0^\infty(\mathbb{R}^{n+1})$ in the norm

$$\| u \|_{W_{p,V}^{1,2}(\mathbb{R}^{n+1})} = \| u \|_{W_p^{1,2}(\mathbb{R}^{n+1})} + \| Vu \|_{L^p(\mathbb{R}^{n+1})} .$$

Proof of Theorem 4.1. By Lemmas 4.1 and 4.2, we get

$$\begin{aligned} & \| D^2 u \|_{L^p(\mathbb{R}^{n+1})} \\ \leq\ & C \| Au \|_{L^p(\mathbb{R}^{n+1})} \\ \leq\ & C \{ \| Au \|_{L^p(\mathbb{R}^{n+1})} + \| Du \|_{L^p(\mathbb{R}^{n+1})} + \| u \|_{L^p(\mathbb{R}^{n+1})} \} \\ \leq\ & C \{ \| Lu \|_{L^p(\mathbb{R}^{n+1})} + \| Vu \|_{L^p(\mathbb{R}^{n+1})} + \| Du \|_{L^p(\mathbb{R}^{n+1})} + \| u \|_{L^p(\mathbb{R}^{n+1})} \} \end{aligned}$$

and

$$\| Vu \|_{L^p(\mathbb{R}^{n+1})} \leq C \left\{ \| Lu \|_{L^p(\mathbb{R}^{n+1})} + \| Du \|_{L^p(\mathbb{R}^{n+1})} + \| u \|_{L^p(\mathbb{R}^{n+1})} \right\} .$$

Applying the two inequalities above we have

$$\begin{aligned} & \| u \|_{W_p^{1,2}(\mathbb{R}^{n+1})} + \| Vu \|_{L^p(\mathbb{R}^{n+1})} \\ \leq\ & C \left\{ \| Lu \|_{L^p(\mathbb{R}^{n+1})} + \| Du \|_{L^p(\mathbb{R}^{n+1})} + \| u \|_{L^p(\mathbb{R}^{n+1})} \right\} . \end{aligned}$$

Then, by the classical interpolation inequality (see [15])

$$\| Du \|_{L^p(\mathbb{R}^{n+1})} \leq \varepsilon \| D^2 u \|_{L^p(\mathbb{R}^{n+1})} + \frac{C}{\varepsilon} \| u \|_{L^p(\mathbb{R}^{n+1})}$$

and taking $\varepsilon = \frac{1}{2C}$, we get

$$\| u \|_{W_p^{1,2}(\mathbb{R}^{n+1})} + \| Vu \|_{L^p(\mathbb{R}^{n+1})} \leq C \left\{ \| Lu \|_{L^p(\mathbb{R}^{n+1})} + \| u \|_{L^p(\mathbb{R}^{n+1})} \right\} . \qquad \square$$

Finally, we give the second main result in this section.

Theorem 4.2. *Suppose that the operator L satisfies* (1.1), (1.3) *and* (1.6), *then there exist λ_0, $C > 0$ such that for any $1 < p \le q$, $\lambda \ge \lambda_0$, $u \in C_0^\infty(\mathbb{R}^{n+1})$ the following estimate holds*:

$$\| u \|_{W^{1,2}_{p,V}(\mathbb{R}^{n+1})} \le C \| Lu + \lambda u \|_{L^p(\mathbb{R}^{n+1})} .$$

Proof. By Theorem 4.1, we have

$$\| u \|_{W^{1,2}_p(\mathbb{R}^{n+1})} + \| Vu \|_{L^p(\mathbb{R}^{n+1})}$$
$$\le C \left\{ \| Lu + \lambda u \|_{L^p(\mathbb{R}^{n+1})} + (\lambda + 1) \| u \|_{L^p(\mathbb{R}^{n+1})} \right\} .$$

If $\lambda \ge \lambda_0 \ge 1$, then $\lambda + 1 \le 2\lambda$ and it is seen that to prove Theorem 4.2, it suffices to prove that

$$\lambda \| u \|_{L^p(\mathbb{R}^{n+1})} \le C \| Lu + \lambda u \|_{L^p(\mathbb{R}^{n+1})} . \tag{4.2}$$

We use a technique suggested by S. Agmon to get the above estimate from Theorem 4.1. Consider the space

$$\mathbb{R}^{n+2} = \{(z,t) = (x,y,t) : t, y \in \mathbb{R}, x \in \mathbb{R}^n\}$$

and the function

$$\tilde{u}(z,t) = u(x,t)\zeta(y)\cos(\mu y),$$

where $\mu = \sqrt{\lambda}$ and ζ is a $C_0^\infty(\mathbb{R})$ function, $\zeta \not\equiv 0$. And introduce the operator

$$\tilde{L}\nu(z,t) = L(x,t)\nu(z,t) - \nu_{yy}(z,t).$$

It is easy to note that we can apply Theorem 4.1 to the operator $\tilde{L}$ in $\mathbb{R}^{n+1}$ and the function $\tilde{u}$. And we get

$$\| \tilde{u}_{zz} \|_{L^p(\mathbb{R}^{n+2})} \le C_1^{-1} \left(\| \tilde{L}\tilde{u} \|_{L^p(\mathbb{R}^{n+2})} + \| \tilde{u} \|_{L^p(\mathbb{R}^{n+2})} \right). \tag{4.3}$$

Now, since $\nu \ge 1$, we then have

$$\int_{\mathbb{R}} |\zeta(y)\cos(\mu y)|^p dy \ge C_1 > 0,$$

where the constant C_1 is independent of ν. Hence,

$$\| u \|^p_{L^p(\mathbb{R}^{n+1})} \le C_1^{-1}\mu^{-2p} \left(\int_{\mathbb{R}} |\zeta(y)\cos(\mu y)|^p dy \right)^{-1} \int_{\mathbb{R}^{n+2}} \Big| \tilde{u}_{yy}(z,t)$$
$$- u(x,t)\Big(\zeta''(y)\cos(\mu y) - 2\mu\zeta'(y)\sin(\mu y) \Big) \Big|^p dzdt$$
$$\le C\mu^{-2p} \left(\| \tilde{u}_{zz} \|^p_{L^p(\mathbb{R}^{n+2})} + (\mu^p + 1) \| u \|^p_{L^p(\mathbb{R}^{n+1})} \right).$$

This and (4.3) yield

$$\mu^2 \| u \|_{L^p(\mathbb{R}^{n+1})} \le C \| \tilde{L}\tilde{u} \|_{L^p(\mathbb{R}^{n+2})} + C(\mu + 1) \| u \|_{L^p(\mathbb{R}^{n+1})} .$$

Since

$$\tilde{L}\tilde{u} = \zeta(y)\cos(\mu y)(Lu + \lambda u) + u[2\mu\zeta'\sin(\mu y) - \zeta''(y)\cos(\mu y)],$$

we have

$$\| \tilde{L}\tilde{u} \|_{L^p(\mathbb{R}^{n+2})} \leq C \| Lu + \lambda u \|_{L^p(\mathbb{R}^{n+1})} + C(\mu+1) \| u \|_{L^p(\mathbb{R}^{n+1})} .$$

So that

$$\lambda \| u \|_{L^p(\mathbb{R}^{n+1})} \leq C_2 \| Lu + \lambda u \|_{L^p(\mathbb{R}^{n+1})} + C_3(\sqrt{\lambda}+1) \| u \|_{L^p(\mathbb{R}^{n+1})},$$

where C_2, C_3 are constants. For $\lambda \geq \lambda_0 = 16{C_3}^2 + 4N_3$ we have $N_3\sqrt{\lambda} \leq \frac{1}{4}\lambda$, $N_3 \leq \frac{1}{4}\lambda$, $N_3(\sqrt{\lambda}+1) \leq \frac{1}{2}\lambda$ and we get 4.2 with $C = 2C_2$. The proof is complete. □

Remark 4.1. Lemmas 4.1 and 4.2, Theorems 4.1 and 4.2 still hold if the condition $a_{ij}(x') \in VMO$ is replaced by the following condition, that is, for sufficiently small ϵ, there exists $R > 0$ such that

$$\sup_{x_0' \in \mathbb{R}^n, r \leq R} \frac{1}{|B(x_0', r)|} \int_{B(x_0', r)} |a_{ij}(y') - (a_{ij})_B| dy' < \varepsilon.$$

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