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STABILIZED-PENALIZED COLLOCATED FINITE VOLUME SCHEME FOR INCOMPRESSIBLE BIOFLUID FLOWS

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Abstract. In this paper, a stabilized-penalized collocated finite volume (SPCFV) scheme is developed and studied for the stationary generalized Navier-Stokes equations with mixed Dirichlet-traction boundary conditions modelling an incompressible biological fluid flow. This method is based on the lowest order approximation (piecewise constants) for both velocity and pressure unknowns. The stabilization-penalization is performed by adding discrete pressure terms to the approximate formulation. These simultaneously involve discrete jump pressures through the interior volume-boundaries and discrete pressures of volumes on the domain boundary. Stability, existence and uniqueness of discrete solutions are established. Moreover, a convergence analysis of the nonlinear solver is also provided. Numerical results from model tests are performed to demonstrate the stability, optimal convergence in the usual L^2 and discrete H^1 norms as well as robustness of the proposed scheme with respect to the choice of the given traction vector.

1. Introduction

The Navier-Stokes equations have been extensively applied by numerous researchers to model most incompressible processes in biofluid mechanics. Despite this, the choice of appropriate boundary conditions remains a challenging and open issue. On the one hand, an ideal method would allow the flow and any information carried with it to exit the domain without adverse upstream effects [25]. On the other hand, it should allow for stable computations regardless of the location of the outflow boundary, even at high Reynolds numbers. The classical use of Dirichlet boundary conditions has led to a relatively well elaborated theory. However, the Dirichlet boundary conditions are not natural in many situations occurring for biological fluid flows where the output of a long channel (vessel) can be important. A Dirichlet boundary condition may be used on the inlet boundary and on the fixed walls of the channel, but it cannot be prescribed on an assumed outlet because the velocity on the outlet depends

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on the flow in the whole channel and it is not known in advance. Furthermore, introducing proper inflow and outflow boundary conditions in numerical computations would allow us to truncate a big physical domain and use a reduced computational domain to make the flow simulation problem more tractable. Several techniques for dealing with outflow boundary conditions exist for incompressible flows (see [25] for a review up to the mid-1990s).

In the presence of artificial boundaries through which the fluid may enter or leave the domain, there is no general agreement on the kind of conditions on these boundaries are the most appropriate on the modeling point of view. Indeed, the different boundary conditions describe the different physical phenomena, and the ability of the artificial conditions to correctly represent the real unbounded domain is crucial for the accuracy of the computed flow field in the context of an incompressible fluid. These conditions may greatly influence the flow inside the computational domain, since any error on the flow field at the boundary may be instantaneously propagated in the whole domain.

Some authors, dealing mostly with numerical methods, use the much less intrusive boundary condition:

$$\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = \mathbf{s}. \tag{1.1}$$

This type of boundary conditions is commonly used since the left-hand side of (1.1) naturally appears in the derivation of the standard variational formulation related to the Navier-Stokes equations. The so-called "Do nothing" boundary condition (1.1) leads to a well-posed problem and work well on numerical experiments. Nevertheless, it is based on a reference flow velocity and a reference pressure, and such data have to be well chosen. It is worth noting that such a boundary condition does not have a genuine physical meaning.

The problem we are presently interested in is the computation of a flow whose velocity is prescribed at one part of the domain boundary and it flows freely on the other one according to the boundary condition (1.1). In this framework, we are often required to truncate the physical domain to obtain a reduced computational domain, either because we want to save computational resources or because the physical domain is unbounded. None of these boundary conditions on the output of a channel excludes the possibility of backward flows that could eventually bring an uncontrollable amount of kinetic energy back to the channel. Consequently, these boundary conditions do not enable us to derive the energy inequality, known from the theory of the Navier-Stokes equations with the no-slip boundary condition, or another equivalent a priori estimate of a weak solution. Due to this fact, the issue of the global existence of a weak solution of the Navier-Stokes equations in a channel with one of these boundary conditions on the output is still open.

Various numerical schemes for solving the Stokes and Navier-Stokes equations have been extensively studied (see [3, 10, 13–16, 22–24, 26] and the references therein). Among these, finite difference (FD), finite element (FE) and finite volume (FV) schemes are frequently used in mathematical or engineering studies. Generally speaking, FV schemes are treated as efficient middle grounds between FD and FE schemes, and therefore rigorously developed by attempting to use FE ideas. Another advantage of FV schemes is that the unknowns can be approximated by piecewise constant functions. This makes it easy to take into account additional nonlinear phenomena. The use of collocated finite volumes for fluid flow problems is appealing for several reasons. Among these, let us mention an inexpensive assembling step compared to finite elements, because there is no numerical integration to perform, the possibility to use, at least to some extent, general unstructured meshes with a low complexity of the data structure (compared with staggered schemes) suitable for the implementation of adaptative mesh refinement strategies and, finally, an easy coupling with additional conservation laws solvers, when these latters are developed within the FV framework. These features make collocated FV schemes attractive for industrial problems, and they are widely used in CFD. When applied to incompressible flow problems, cell-centered collocated FV schemes seem to suffer from a lack of coercivity, which was shown in [11] to be cured by a stabilization technique similar to those used in the FE context. However, for high Reynolds number flows and for computationally reasonable meshes, the amount of stabilization necessary to damp pressure oscillations was often found to be the cause of severe accuracy degradation. Finally, regardless of their physical interpretations, FV schemes can be mathematically treated as Petrov-Galerkin methods with trial function spaces associated with certain finite element spaces and test spaces related to finite volumes.

The aim of the present paper is to design and analyze a collocated finite volume approximation of the steady incompressible Navier-Stokes problem with Dirichlet conditions on one part of the domain boundary and boundary conditions of (1.1)-type on the remaining part. The considered computational domain may be either an originally bounded region with apertures or the bounded truncation of a larger physical region by means of artificial boundaries. The so-called SPCFV scheme is developed as a Petrov-Galerkin method through its relationship with the lowest equal-order mixed FE pair Q_0-Q_0. Moreover, owing to the known instability of the latter in the finite element context, stabilizing pressure-jump terms through the interior boundaries between the volumes are cautiously added into the discrete formulation. Likewise, penalizing terms involving discrete pressures of volumes on the domain boundary are also introduced. Through estimates of discrete velocity and pressure solutions, the stability is achieved for both linear and nonlinear cases. Then, the existence and uniqueness of discrete solutions are derived. A rigorous error analysis is not yet available and presently under investigation. However, some results concerning the convergence rates of the SPCFV scheme are reported and discussed in the numerical tests.

The paper is organized as follows. In the next section, we present the considered continuous problem and the weak formulations which are used in the subsequent analysis. Moreover, assumptions on the data are made and definitions of function spaces are introduced. In Section 3, we define the discretization spaces and recall some fundamental results on the finite volume schemes. The stabilized-penalized collocated finite volume (SPCFV) scheme is then presented and analyzed for the linear Stokes problem and the nonlinear Navier-Stokes equations in Section 4. For illustrating the efficiency of the SPCFV scheme, some numerical results are presented in Section 5. Finally, certain conclusions are drawn in the closing section.

2. Mathematical setting

Let $\Omega \subset \mathbb{R}^2$ be a polygonal open bounded domain with a regular boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ such that $\Gamma_D \cap \Gamma_N = \emptyset$ and $\Gamma_D, \Gamma_N \neq \emptyset$. As discussed above, Ω can also represent some bounded truncation of a much larger unbounded region in $\mathbb{R}^2$. Lebesgue measure and diameter of Ω in $\mathbb{R}^2$ will be denoted by $|\Omega|$ and $\delta(\Omega)$, respectively. We are interested in finding the velocity vector field $\mathbf{u} : \Omega \to \mathbb{R}^2$ and the pressure scalar field $p : \Omega \to \mathbb{R}$ which satisfy the generalized Navier-Stokes problem:

$$\eta\mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_D, \tag{2.3}$$

$$\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\mathbf{n} = \mathbf{s} \quad \text{on } \Gamma_N. \tag{2.4}$$

Here, $\nu > 0$ is the kinematic viscosity coefficient (inversely proportional to the Reynolds number Re), $\mathbf{f}$ a given external body force, $\mathbf{g}$ a prescribed (injection) velocity on the boundary part Γ_D, $\mathbf{s}$ a prescribed traction vector on the boundary part Γ_N, $\mathbf{n}$ the outward-pointing normal to the boundary and $\frac{\partial\mathbf{u}}{\partial\mathbf{n}}$ denotes the normal-directional derivative. Moreover, $\eta \geq 0$ is a real parameter that may come from the time discretization of the evolution term $\frac{\partial\mathbf{u}}{\partial t}$ in the unsteady-state Navier-Stokes equations (*cf.* [11]). Since $\Gamma_N \neq \emptyset$, it is widely admitted that the outflow condition (2.4) ensures the uniqueness of the pressure.

We also make the following assumptions:

(H1) $\mathbf{f} \in \mathbf{L}^2(\Omega) = [L^2(\Omega)]^2$.
(H2) $\mathbf{g} \in \mathbf{L}^2(\Gamma_D) = [L^2(\Gamma_D)]^2$.
(H3) $\mathbf{s} \in \mathbf{L}^2(\Gamma_N) = [L^2(\Gamma_N)]^2$.

For the sake of convenience and without loss of generality, we may as well suppose that $\mathbf{g} = \mathbf{0}$. Nonhomogeneous Dirichlet boundary conditions can also be taken as it will be the case below in the numerical experiments.

Now, let us define the standard weak formulation of (2.1)-(2.4). First, the function spaces we deal with are as follows:

$$\mathbf{V} = \{\mathbf{v} = (v^{(1)}, v^{(2)}) \in [H^1(\Omega)]^2;\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}, \tag{2.5}$$

$$Q = L^2(\Omega), \tag{2.6}$$

endowed with the usual L^2-type scalar products and associated norms.

The weak formulation of the steady generalized Navier-Stokes problem (2.1)-(2.4) consists to find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(p,\mathbf{v}) &= (\mathbf{f},\mathbf{v})_\Omega + (\mathbf{s},\mathbf{v})_{\Gamma_N} && \forall \mathbf{v} \in \mathbf{V}, \\ -\,b(q,\mathbf{u}) &= 0 && \forall q \in Q, \end{aligned} \tag{2.7}$$

where the forms $a(\cdot,\cdot) : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$, $c(\cdot,\cdot,\cdot) : \mathbf{V} \times \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ and $b(\cdot,\cdot) : Q \times \mathbf{V} \to \mathbb{R}$ are defined for all $\mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathbf{V}$ and $q \in Q$ by

$$\begin{cases} a(\mathbf{w},\mathbf{v}) = \eta(\mathbf{w},\mathbf{v})_\Omega + \nu \displaystyle\sum_{i=1}^{2} (\nabla w^{(i)}, \nabla v^{(i)})_\Omega, \\ c(\mathbf{u},\mathbf{w},\mathbf{v}) = ((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})_\Omega = \displaystyle\sum_{i,j=1}^{2} \int_\Omega u^{(j)} \frac{\partial w^{(i)}}{\partial x_j} v^{(i)}\, dx, \\ b(q,\mathbf{v}) = -(q, \operatorname{div}\mathbf{v})_\Omega, \end{cases} \tag{2.8}$$

with $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{\Gamma_N}$ being the usual inner products on $L^2(\Omega)$, $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^2(\Gamma_N)$ accordingly.

Existence and uniqueness of a solution to (2.7), also called weak solution of (2.1)-(2.4), are addressed in many monographs and papers (see, e.g., [1, 2, 4, 5, 12, 14, 17, 18, 20, 21]). In the remainder of the paper, we need assume the existence of at least one solution to (2.7).

In dealing with the problems (2.1)-(2.4), we also need analyze the related linear problem, namely the generalized steady Stokes problem, which reads as:

$$\eta\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{2.9}$$

supplemented by (2.2)-(2.4). Similarly to (2.7), the weak formulation of this problem consists to find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(p,\mathbf{v}) &= (\mathbf{f},\mathbf{v})_\Omega + (\mathbf{s},\mathbf{v})_{\Gamma_N} && \forall \mathbf{v} \in \mathbf{V}, \\ b(q,\mathbf{u}) &= 0 && \forall q \in Q. \end{aligned} \tag{2.10}$$

Finally, for numerical matters introduce the compacted bilinear form

$$\mathcal{B}((\mathbf{w}, r); (\mathbf{v}, q)) = a(\mathbf{w},\mathbf{v}) + b(r,\mathbf{v}) - b(q,\mathbf{w}) \tag{2.11}$$

and linear form

$$\mathcal{L}(\mathbf{v}, q) = (\mathbf{f},\mathbf{v})_\Omega + (\mathbf{s},\mathbf{v})_{\Gamma_N}. \tag{2.12}$$

In doing so, weak formulations (2.10) and (2.7) can be recast as: find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\mathcal{B}((\mathbf{u}, r); (\mathbf{v}, q)) = \mathcal{L}(\mathbf{v}, q) \qquad \forall (\mathbf{v}, q) \in \mathbf{V} \times Q \tag{2.13}$$

and

$$\mathcal{B}((\mathbf{u}, r); (\mathbf{v}, q)) + c(\mathbf{u},\mathbf{u},\mathbf{v}) = \mathcal{L}(\mathbf{v}, q) \qquad \forall (\mathbf{v}, q) \in \mathbf{V} \times Q, \tag{2.14}$$

respectively.

3. Discretization aspects

3.1. Domain discretization

First, let us present the notion of admissible discretization of the domain Ω which will be necessary for introducing the SPCFV scheme.

Definition 3.1. An admissible finite volume mesh of Ω, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ where:

- $\mathcal{M}$ is a finite family of disjoint non-empty convex subdomains K of Ω (called control volumes) such that:
  - each control volume is either a rectangle or a triangle with the interior angles strictly lower than $\pi/2$;
  - $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K}\backslash K$ be the boundary of K and $|K| > 0$ denotes the measure of K;
- $\mathcal{E}$ is a finite family of disjoint subsets σ of $\overline{\Omega}$ (called mesh edges) such that, for all $\sigma \in \mathcal{E}$, there exist a hyper-plane E of $\mathbb{R}^2$ and $K \in \mathcal{M}$ with $\overline{\sigma} = \partial K \cap E$ and σ is non-empty open subset of E with the two-dimensional measure $|\sigma| > 0$. We assume that, for any $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. Furthermore $\mathcal{E} = \cup_{K\in\mathcal{M}}\mathcal{E}_K$. We then assume that $\mathcal{E}$ is partitioned into $\mathcal{E} = \mathcal{E}_{int} \cup \mathcal{E}_{ext}$, such that:
  - $\mathcal{E}_{int} = \{\sigma \in \mathcal{E}; \sigma \not\subset \partial\Omega\}$ (set of interior edges);
  - $\mathcal{E}_{ext} = \{\sigma \in \mathcal{E}; \sigma \subset \partial\Omega\}$ (set of boundary edges);
    - $\mathcal{E}_{extD} = \{\sigma \in \mathcal{E}_{ext}; \sigma \subset \Gamma_D\}$;
    - $\mathcal{E}_{extN} = \{\sigma \in \mathcal{E}_{ext}; \sigma \subset \Gamma_N\}$;
- $\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$ and denoted by $\mathcal{P} = (x_K)_{K\in\mathcal{M}}$ (called collocation points), which are chosen such that x_K is the intersection of the perpendicular bisectors of each edge.

Note that any internal edge σ separating two control volumes K and L is denoted by $\sigma = K|L$ and satisfies the orthogonality condition:

$$x_\sigma = [x_K, x_L] \cap \sigma.$$

The following notations are also used: $(K, L \in \mathcal{M}$ and $\sigma \in \mathcal{E})$

* $d_{K,L}$: Euclidean distance between x_K and x_L,
* $d_{K,\sigma}$: Euclidean distance between x_K and x_σ,
* m_K : measure of the control volume K,
* m_σ : measure of the edge σ,
* h_K : diameter of each control volume K,
* $h = \max_{K\in\mathcal{M}} h_K$: mesh parameter,
* $\mathcal{N}_K$: subset of $\mathcal{M}$ of neighboring control volumes to K (i.e., elements in $\mathcal{M}$ sharing an edge with K) excluding K.

In order to unify notation, we also use

$$d_\sigma = \begin{cases} d_{K,L} & \text{if} \quad \sigma = K|L \in \mathcal{E}_{int}, \\ d_{K,\sigma} & \text{if} \quad \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K. \end{cases}$$

The regularity of the mesh $\mathcal{D}$ is measured via the parameter:

$$\begin{aligned} regul(\mathcal{D}) = \; \inf \Big\{ & \Big\{ \frac{d_{K,\sigma}}{h_K};\; K \in \mathcal{M},\, \sigma \in \mathcal{E}_K \Big\} \cup \Big\{ \frac{h_K}{h};\; K \in \mathcal{M} \Big\} \\ & \cup \Big\{ \frac{d_{K,\sigma}}{d_\sigma};\; K \in \mathcal{M},\, \sigma = K|L \in \mathcal{E}_K \Big\} \cup \Big\{ \frac{1}{Card(\mathcal{E}_K)};\; K \in \mathcal{M} \Big\} \Big\}. \end{aligned} \tag{3.1}$$

3.2. Discrete functional setting

Definition 3.2. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be an admissible discretization of Ω in the sense of Definition 3.1 with a constant θ selected so that $regul(\mathcal{D}) > \theta > 0$. We define $H_\mathcal{D}(\Omega) \subset L^2(\Omega)$ to be the space of functions which are piecewise constant over each control volume $K \in \mathcal{M}$. For all $v \in H_\mathcal{D}(\Omega)$ and $K \in \mathcal{M}$, v_K will denote the constant value of v in K. Thus, each $v \in H_\mathcal{D}(\Omega)$ will be identified with the family of its values $(v_K)_{K\in\mathcal{M}}$ on the control volumes. We also set: $\mathbf{E}_\mathcal{D}(\Omega) = [H_\mathcal{D}(\Omega)]^2$.

It is important to note that a given function $v \in H_\mathcal{D}(\Omega)$, considered only as an element of $L^2(\Omega)$, would not have a trace on $\partial\Omega$. However, since such a v is constant on the control volumes, we can set $v_{|\sigma} = v_K$ for any edge $\sigma \in \mathcal{E}_{ext}$ with $K \in \mathcal{M}$ being the unique volume such that $\sigma \in \mathcal{E}_K$.

Now, introduce on $H_\mathcal{D}(\Omega)$ the following inner products:

$$[w, v]_{1,\mathcal{D}} = \sum_{\substack{\sigma\in\mathcal{E}_{int} \\ (\sigma=K|L)}} \frac{m_\sigma}{d_\sigma} (w_L - w_K)\,(v_L - v_K) + \sum_{\substack{\sigma\in\mathcal{E}_{extD} \\ (\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma} w_K\, v_K, \tag{3.2}$$

$$\begin{aligned} \langle w, v\rangle_\mathcal{D} = & \sum_{\substack{\sigma\in\mathcal{E}_{int} \\ (\sigma=K|L)}} \frac{m_\sigma}{d_\sigma} (h_K^2 + h_L^2)(w_L - w_K)(v_L - v_K) \\ & + \sum_{\substack{\sigma\in\mathcal{E}_{ext} \\ (\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma}\, h_K^2\, w_K\, v_K \end{aligned} \tag{3.3}$$

and the associated norms: $\|w\|_{1,\mathcal{D}} = \sqrt{[w,w]_{1,\mathcal{D}}}$, $\|w\|_\mathcal{D} = \sqrt{\langle w, w\rangle_\mathcal{D}}$ for any $w, v \in H_\mathcal{D}(\Omega)$. In relation with (3.3), we will also make use of the discrete stabilizing-penalizing form defined for any $\lambda, \gamma > 0$ by

$$\begin{aligned} \langle q, r\rangle_{\mathcal{D},\lambda,\gamma} = \lambda & \sum_{\substack{\sigma\in\mathcal{E}_{int} \\ (\sigma=K|L)}} \frac{m_\sigma}{d_\sigma} (h_K^2 + h_L^2)(q_L - q_K)(r_L - r_K) \\ & + \gamma \sum_{\substack{\sigma\in\mathcal{E}_{ext} \\ (\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma}\, h_K^2\, q_K\, r_K \end{aligned} \tag{3.4}$$

for any $q, r \in H_{\mathcal{D}}(\Omega)$.

Similarly, on $\mathbf{E}_{\mathcal{D}}(\Omega)$ we only need define the inner product and associated norm:

$$[\mathbf{w}, \mathbf{v}]_{1,\mathcal{D}} = \sum_{\substack{\sigma\in\mathcal{E}_{int} \\ (\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(\mathbf{w}_L - \mathbf{w}_K)\cdot(\mathbf{v}_L - \mathbf{v}_K) + \sum_{\substack{\sigma\in\mathcal{E}_{extD} \\ (\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma}\mathbf{w}_K\cdot\mathbf{v}_K \tag{3.5}$$

and $\|\mathbf{w}\|_{1,\mathcal{D}} = \sqrt{[\mathbf{w}, \mathbf{w}]_{1,\mathcal{D}}}$ for any $\mathbf{w}, \mathbf{v} \in \mathbf{E}_{\mathcal{D}}(\Omega)$.

Analogously to the continuous case, we may also get the discrete Poincare-type inequalities as follows.

Lemma 3.1. *We have*

$$\|w\|_{\boldsymbol{L}^2(\Omega)} \leq \delta(\Omega)\, \|w\|_{1,\mathcal{D}} \qquad \forall w \in H_{\mathcal{D}}(\Omega), \tag{3.6}$$

$$\|q\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{\delta(\Omega)}{\theta h\sqrt{2}}\|q\|_{\mathcal{D}} \qquad \forall q \in H_{\mathcal{D}}(\Omega). \tag{3.7}$$

Proof. A detailed proof of (3.6) is given in [10]. Next, let $q \in H_{\mathcal{D}}(\Omega)$. From (3.3), we get

$$\begin{aligned} \|q\|_{\mathcal{D}}^2 &= \sum_{\substack{\sigma\in\mathcal{E}_{int} \\ (\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(h_K^2 + h_L^2)(q_L - q_K)^2 + \sum_{\substack{\sigma\in\mathcal{E}_{ext} \\ (\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma}\, h_K^2\, q_K^2 \\ &\geq 2 \min_{K\in\mathcal{M}} h_K^2 \left[\sum_{\substack{\sigma\in\mathcal{E}_{int} \\ (\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(q_L - q_K)^2 + \sum_{\substack{\sigma\in\mathcal{E}_{ext} \\ (\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma} q_K^2 \right] \\ &\geq 2\,\theta^2 h^2 \|q\|_{1,\mathcal{D}}^2. \end{aligned}$$

Thus

$$\|q\|_{1,\mathcal{D}} \leq \frac{1}{\theta h\sqrt{2}}\|q\|_{\mathcal{D}}$$

whose combination with (3.6) gives (3.7). □

4. Approximation of the generalized Navier-Stokes problem

Let $\mathcal{D}$ be an admissible rectangular discretization of a connected polygonal domain $\Omega \subset \mathbb{R}^2$ in the sense of Definition 3.1 and select a constant θ so that $regul(\mathcal{D}) > \theta > 0$. Moreover, we assume that $\mathcal{D}$ is quasi-uniform in the following sense:

$$d_{K,\sigma} = d_{L,\sigma} \qquad \forall \sigma = K|L \in \mathcal{E}_{int}. \tag{4.1}$$

First, let us start by introducing a discrete divergence operator $div_{\mathcal{D}}$: $\mathbf{E}_{\mathcal{D}}(\Omega) \to H_{\mathcal{D}}(\Omega)$ defined locally by

$$(div_{\mathcal{D}}\mathbf{v})_{|K} = \sum_{\substack{L\in\mathcal{N}_K \\ (\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(d_{L,\sigma}\mathbf{v}_K + d_{K,\sigma}\mathbf{v}_L)\cdot\mathbf{n}_{K,\sigma} + \sum_{\substack{\sigma\in\mathcal{E}_{extN} \\ (\sigma\in\mathcal{E}_K)}} m_\sigma(\mathbf{v}_K\cdot\mathbf{n}_\sigma) \tag{4.2}$$

for all $K \in \mathcal{M},\ \mathbf{v} \in \mathbf{E}_{\mathcal{D}}(\Omega)$.

4.1. Approximation of the linear problem

Before handling the generalized steady Navier-Stokes problem (2.7), let us address the related linear problem (2.10). Results of this part will be of critical use in the sequel.

It is natural to approximate the mixed formulation (2.10) with the discrete problem which consists in finding $(\mathbf{u}_h, p_h) \in \mathbf{E}_{\mathcal{D}}(\Omega) \times H_{\mathcal{D}}(\Omega)$ such that

$$\begin{aligned} \eta(\mathbf{u}_h, \mathbf{v})_\Omega + \nu[\mathbf{u}_h, \mathbf{v}]_{1,\mathcal{D}} - (p_h, div_{\mathcal{D}}\mathbf{v})_\Omega &= (\mathbf{f}, \mathbf{v})_\Omega + (\mathbf{s}, \mathbf{v})_{\Gamma_N} \ \forall \mathbf{v} \in \mathbf{E}_{\mathcal{D}}(\Omega), \\ (q, div_{\mathcal{D}}\mathbf{u}_h)_\Omega + \langle p_h, q\rangle_{\mathcal{D},\lambda,\gamma} &= 0 \ \forall q \in H_{\mathcal{D}}(\Omega), \end{aligned} \tag{4.3}$$

where $\langle p, q\rangle_{\mathcal{D},\lambda,\gamma}$ is the stabilizing-penalizing term needed because of the collocated type of the velocity and pressure approximations; λ, γ being adjustable positive parameters which will be tuned in order to enhance stability. It is also important to note that the Dirichlet boundary condition is already imposed inside the inner product $[\mathbf{u}_h, \mathbf{v}]_{1,\mathcal{D}}$ (see (3.2)).

Here again, by defining the compacted bilinear form:

$$\begin{aligned} \mathcal{B}_h((\mathbf{w}, r); (\mathbf{v}, q)) = \eta(\mathbf{w}, \mathbf{v})_\Omega + \nu[\mathbf{w}, \mathbf{v}]_{1,\mathcal{D}} - (r, div_{\mathcal{D}}\mathbf{v})_\Omega \\ + (q, div_{\mathcal{D}}\mathbf{w})_\Omega + \langle r, q\rangle_{\mathcal{D},\lambda,\gamma} \end{aligned} \tag{4.4}$$

the stabilized-penalized discrete formulation (4.3) can be rewritten in the equivalent form: find $(\mathbf{u}_h, p_h) \in \mathbf{E}_{\mathcal{D}}(\Omega) \times H_{\mathcal{D}}(\Omega)$ such that

$$\mathcal{B}_h((\mathbf{u}_h, p_h; (\mathbf{v}, q)) = \mathcal{L}(\mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in \mathbf{E}_{\mathcal{D}}(\Omega) \times H_{\mathcal{D}}(\Omega). \tag{4.5}$$

Next, let us present an important result of stability estimates for both discrete velocity and pressure solutions of (4.3).

Lemma 4.1 (Estimates for the discrete velocity and pressure)**.** *Under all above hypotheses, let also assume that formulation* (4.3) (*or* (4.5)) *admits a solution* $(\mathbf{u}_h, p_h) \in \mathbf{E}_{\mathcal{D}}(\Omega) \times H_{\mathcal{D}}(\Omega)$. *Then, the following estimates hold*:

$$\begin{aligned} &\eta \,\|\mathbf{u}_h\|^2_{\mathbf{L}^2(\Omega)} + \frac{\nu}{2}\|\mathbf{u}_h\|^2_{1,\mathcal{D}} + \min\{\lambda, \gamma\}\,\|p_h\|^2_{\mathcal{D}} \\ &\le \frac{[\delta(\Omega)]^2}{\nu}\left(\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{s}\|^2_{\mathbf{L}^2(\Gamma_N)}\right), \end{aligned} \tag{4.6}$$

$$\min\{\lambda, \gamma\}\, h^2\,\|p_h\|^2_{L^2(\Omega)} \le \frac{[\delta(\Omega)]^4}{2\nu\,\theta^2}\left(\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{s}\|^2_{\mathbf{L}^2(\Gamma_N)}\right). \tag{4.7}$$

Proof. Let us set $\mathbf{v} = \mathbf{u}_h$ and $q = p_h$ in (4.3) to obtain the two equations

$$\begin{cases} \eta\,\|\mathbf{u}_h\|^2_{\mathbf{L}^2(\Omega)} + \nu\|\mathbf{u}_h\|^2_{1,\mathcal{D}} - (p_h, div_{\mathcal{D}}\mathbf{u}_h)_\Omega = (\mathbf{f}, \mathbf{u}_h)_\Omega + (\mathbf{s}, \mathbf{u}_h)_{\Gamma_N}, \\ \qquad (p_h, div_{\mathcal{D}}\mathbf{u}_h)_\Omega + \langle p_h, p_h\rangle_{\mathcal{D},\lambda,\gamma} = 0. \end{cases}$$

Substituting for $(p_h, div_{\mathcal{D}}\mathbf{u}_h)_\Omega$ from the second equation into the first one gives

$$\eta\,\|\mathbf{u}_h\|^2_{\mathbf{L}^2(\Omega)} + \nu\|\mathbf{u}_h\|^2_{1,\mathcal{D}} + \langle p_h, p_h\rangle_{\mathcal{D},\lambda,\gamma} = (\mathbf{f}, \mathbf{u}_h)_\Omega + (\mathbf{s}, \mathbf{u}_h)_{\Gamma_N}. \tag{4.8}$$

On the other hand, Young inequality with $\epsilon=\frac{[\delta(\Omega)]^2}{\nu}$, (3.6) and the well-known trace inequality (see [14]) yield

$$(\mathbf{f},\mathbf{u}_h)_\Omega \leq \frac{[\delta(\Omega)]^2}{\nu}\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\nu}{4}\|\mathbf{u}_h\|^2_{1,\mathcal{D}} \tag{4.9}$$

and

$$(\mathbf{s},\mathbf{u}_h)_{\Gamma_N} \leq \frac{[\delta(\Omega)]^2}{\nu}\|\mathbf{s}\|^2_{\mathbf{L}^2(\Gamma_N)} + \frac{\nu}{4}\|\mathbf{u}_h\|^2_{1,\mathcal{D}}. \tag{4.10}$$

It remains to combine (4.8) with (4.9), (4.10) and (3.4) to deduce (4.6). The L^2-estimation (4.7) is a straightforward consequence of (4.6) and (3.7). □

We can now state the existence and uniqueness of a solution to the discrete problem (4.3).

Theorem 4.1 (Existence and uniqueness of a discrete solution)**.** *Under the assumptions of Lemma 4.1, discrete problem* (4.3) (*or* (4.5)) *admits a unique solution* $(\mathbf{u}_h,p_h)\in\mathbf{E}_\mathcal{D}(\Omega)\times H_\mathcal{D}(\Omega)$.

Proof. Let $(\mathbf{u},p)\in\mathbf{E}_\mathcal{D}(\Omega)\times H_\mathcal{D}(\Omega)$ be given. Denote by $(\hat{\mathbf{u}},\hat{p})$ the solution of

$$\begin{cases} (\hat{\mathbf{u}},\mathbf{v})_\Omega = \eta(\mathbf{u},\mathbf{v})_\Omega + \nu[\mathbf{u},\mathbf{v}]_{1,\mathcal{D}} - (p, div_\mathcal{D}\mathbf{v})_\Omega & \forall \mathbf{v}\in\mathbf{E}_\mathcal{D}(\Omega), \\ (\hat{p},q)_\Omega = (q, div_\mathcal{D}\mathbf{u})_\Omega + \langle p,q\rangle_{\mathcal{D},\lambda,\gamma} & \forall q\in H_\mathcal{D}(\Omega). \end{cases} \tag{4.11}$$

We obviously have $(\hat{\mathbf{u}},\hat{p})\in\mathbf{E}_\mathcal{D}(\Omega)\times H_\mathcal{D}(\Omega)$ and we can define the linear mapping $\Psi:\mathbf{E}_\mathcal{D}(\Omega)\times H_\mathcal{D}(\Omega)\to\mathbf{E}_\mathcal{D}(\Omega)\times H_\mathcal{D}(\Omega)$ by $\Psi(\mathbf{u},p)=(\hat{\mathbf{u}},\hat{p})$. Estimate (4.6) implies that if $\Psi(\mathbf{u},p)=(\mathbf{0},0)$, then $(\mathbf{u},p)=(\mathbf{0},0)$. Therefore, $\Psi(\cdot)$ is one-to-one from $\mathbf{E}_\mathcal{D}(\Omega)\times H_\mathcal{D}(\Omega)$ into itself. This completes the proof of existence and uniqueness of a solution $(\mathbf{u}_h,p_h)\in\mathbf{E}_\mathcal{D}(\Omega)\times H_\mathcal{D}(\Omega)$ to problem (4.3) since $(\mathbf{u}_h,p_h)=\Psi(\hat{\mathbf{u}},\hat{p})$ with

$$\hat{\mathbf{u}}_K = \frac{1}{m_K}\left[\int_K \mathbf{f}\,dx + \int_{\partial K\cap\Gamma_N}\mathbf{s}\,d\sigma\right] \quad \text{and} \quad \hat{p}_K = 0 \quad \forall K\in\mathcal{M} \tag{4.12}$$

and these define clearly an element in $\mathbf{E}_\mathcal{D}(\Omega)\times H_\mathcal{D}(\Omega)$. □

4.2. The SPCFV scheme for the nonlinear problem

We can also define for the trilinear form $c(\mathbf{u},\mathbf{v},\mathbf{w})$ the following approximation which can be expressed in two equivalent forms:

$$\begin{aligned} c_\mathcal{D}(\mathbf{u},\mathbf{v},\mathbf{w}) &= \sum_{K\in\mathcal{M}}\left(\sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} m_\sigma(\mathbf{u}_\sigma\cdot\mathbf{n}_{K,\sigma})\mathbf{v}_\sigma + \sum_{\substack{\sigma\in\mathcal{E}_{extN}\\(\sigma\in\mathcal{E}_K)}} m_\sigma(\mathbf{u}_K\cdot\mathbf{n}_\sigma)\mathbf{v}_K\right)\cdot\mathbf{w}_K \\ &\quad - \sum_{K\in\mathcal{M}}(div_\mathcal{D}\mathbf{u})_{|K}\,(\mathbf{v}_K\cdot\mathbf{w}_K) \end{aligned} \tag{4.13}$$

$$= \sum_{K\in\mathcal{M}} \left(\sum_{\substack{\sigma\in\mathcal{E}_{int} \\ (\sigma=K|L)}} m_\sigma (\mathbf{u}_\sigma \cdot \mathbf{n}_{K,\sigma}) \, (\mathbf{v}_\sigma - \mathbf{v}_K) \right) \cdot \mathbf{w}_K$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{E}_\mathcal{D}(\Omega)$. In (4.13) we have set: $\mathbf{u}_\sigma = (\frac{d_{L,\sigma}}{d_\sigma}\mathbf{u}_K + \frac{d_{K,\sigma}}{d_\sigma}\mathbf{u}_L)$ and $\mathbf{v}_\sigma = (\frac{d_{L,\sigma}}{d_\sigma}\mathbf{v}_K + \frac{d_{K,\sigma}}{d_\sigma}\mathbf{v}_L)$.

By virtue of (4.1), we first remark that the trilinear term $c_\mathcal{D}(\cdot,\cdot,\cdot)$ verifies

$$c_\mathcal{D}(\mathbf{w}, \mathbf{v}, \mathbf{v}) = 0 \qquad \forall \mathbf{w}, \mathbf{v} \in \mathbf{E}_\mathcal{D}(\Omega). \tag{4.14}$$

Similarly to the continuous counterpart $c(\cdot,\cdot,\cdot)$, we also have the following boundedness property of $c_\mathcal{D}(\cdot,\cdot,\cdot)$.

Lemma 4.2 (Estimate on $c_\mathcal{D}(\cdot,\cdot,\cdot)$ by discrete H^1 norms)**.** *There exists a positive constant C_1, depending only on Ω and θ, such that*

$$|c_\mathcal{D}(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_1 \|\mathbf{u}\|_{1,\mathcal{D}} \, \|\mathbf{v}\|_{1,\mathcal{D}} \, \|\mathbf{w}\|_{1,\mathcal{D}} \qquad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{E}_\mathcal{D}(\Omega). \tag{4.15}$$

Proof. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{E}_\mathcal{D}(\Omega)$. First, rewrite the quantity $c_\mathcal{D}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ in the following form:

$$\begin{aligned} c_\mathcal{D}(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \sum_{\substack{K\in\mathcal{M}, \\ \sigma=K|L}} m_\sigma \left[(\mathbf{u}_\sigma \cdot \mathbf{n}_{K,\sigma}) \, \mathbf{w}_K \right] \cdot (\mathbf{v}_\sigma - \mathbf{v}_K) \\ &= \sum_{\substack{K\in\mathcal{M}, \\ \sigma=K|L}} \frac{d_{K,\sigma}}{d_\sigma} \frac{m_\sigma}{d_\sigma} \left[\left((d_{L,\sigma}\mathbf{u}_K + d_{K,\sigma}\mathbf{u}_L) \cdot \mathbf{n}_{K,\sigma} \right) \, \mathbf{w}_K \right] \cdot (\mathbf{v}_L - \mathbf{v}_K) \, . \end{aligned}$$

Then, applying the discrete Cauchy-Schwarz inequality twice, Young inequality with $\epsilon = 1/2$ and the fact that

$$\sum_{\substack{L\in\mathcal{N}_K \\ (\sigma=K|L)}} m_\sigma \, d_{K,\sigma} = 2 \, m_K$$

(see [10]), we get successively

$$\begin{aligned} c_\mathcal{D}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq & \left[\sum_{\substack{K\in\mathcal{M}, \\ \sigma=K|L}} \left(\frac{d_{K,\sigma}}{d_\sigma} \right)^2 \frac{m_\sigma}{d_\sigma} \left| d_{L,\sigma}\mathbf{u}_K + d_{K,\sigma}\mathbf{u}_L \right|^2 \, |\mathbf{w}_K|^2 \right]^{1/2} \\ & \times \left[\sum_{\substack{K\in\mathcal{M}, \\ \sigma=K|L}} \frac{m_\sigma}{d_\sigma} \left| \mathbf{v}_L - \mathbf{v}_K \right|^2 \right]^{1/2} \end{aligned}$$

$$
\begin{aligned}
&\leq \sqrt{2}\left[\sum_{\substack{K\in\mathcal{M},\\ \sigma=K|L}} \frac{m_\sigma}{d_\sigma}\left(d_{L,\sigma}^2\left|\mathbf{u}_K\right|^2+d_{K,\sigma}^2\left|\mathbf{u}_L\right|^2\right)\left|\mathbf{w}_K\right|^2\right]^{1/2} \|\mathbf{v}\|_{1,\mathcal{D}} \\
&\leq 2\left[\sum_{\substack{K\in\mathcal{M},\\ \sigma=K|L}} \frac{m_\sigma}{d_\sigma}\, d_{K,\sigma}^2\left|\mathbf{u}_K\right|^2\left|\mathbf{w}_K\right|^2\right]^{1/2} \|\mathbf{v}\|_{1,\mathcal{D}} \\
&\leq 2\left[\sum_{\substack{K\in\mathcal{M},\\ \sigma=K|L}} m_\sigma\, d_{K,\sigma}\left|\mathbf{u}_K\right|^4\right]^{1/4}\left[\sum_{\substack{K\in\mathcal{M},\\ \sigma=K|L}} m_\sigma\, d_{K,\sigma}\left|\mathbf{w}_K\right|^4\right]^{1/4} \|\mathbf{v}\|_{1,\mathcal{D}} \\
&\leq 2\sqrt{2}\left[\sum_{\substack{K\in\mathcal{M},\\ \sigma=K|L}} m_K\left|\mathbf{u}_K\right|^4\right]^{1/4}\left[\sum_{\substack{K\in\mathcal{M},\\ \sigma=K|L}} m_K\left|\mathbf{w}_K\right|^4\right]^{1/4} \|\mathbf{v}\|_{1,\mathcal{D}} \\
&\leq 2\sqrt{2}\, \|\mathbf{u}\|_{\mathbf{L}^4(\Omega)}\, \|\mathbf{w}\|_{\mathbf{L}^4(\Omega)}\, \|\mathbf{v}\|_{1,\mathcal{D}}.
\end{aligned}
$$

The inequality (4.15) is now a straightforward consequence of the following discrete Sobolev inequality (see proof in [6]):

$$
\|v\|_{L^4(\Omega)} \leq C_2\|v\|_{1,\mathcal{D}} \qquad \forall v\in H_{\mathcal{D}}(\Omega), \tag{4.16}
$$

where C_2 depends only on Ω and θ, so that $C_1 = 2\sqrt{2}C_2^2$. □

From this lemma, we can define the positive number:

$$
N = \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{E}_{\mathcal{D}}(\Omega)\setminus\mathbf{0}} \frac{|c_{\mathcal{D}}(\mathbf{u},\mathbf{v},\mathbf{w})|}{\|\mathbf{u}\|_{1,\mathcal{D}}\,\|\mathbf{v}\|_{1,\mathcal{D}}\,\|\mathbf{w}\|_{1,\mathcal{D}}}, \tag{4.17}
$$

which may depend only on Ω and θ.

It is then natural to approximate the mixed formulation (2.7) with the discrete stabilized-penalized problem which consists in finding $(\mathbf{u}_h, p_h)\in\mathbf{E}_{\mathcal{D}}(\Omega)\times H_{\mathcal{D}}(\Omega)$ such that:

$$
\begin{aligned}
&\eta(\mathbf{u}_h,\mathbf{v})_\Omega + \nu[\mathbf{u}_h,\mathbf{v}]_{1,\mathcal{D}} + c_{\mathcal{D}}(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}) \\
&\qquad -(p_h, div_{\mathcal{D}}\mathbf{v})_\Omega = (\mathbf{f},\mathbf{v})_\Omega + (\mathbf{s},\mathbf{v})_{\Gamma_N} \quad \forall\mathbf{v}\in\mathbf{E}_{\mathcal{D}}(\Omega), \\
&(q, div_{\mathcal{D}}\mathbf{u}_h)_\Omega + \langle p_h, q\rangle_{\mathcal{D},\lambda,\gamma} = 0 \qquad \forall q\in H_{\mathcal{D}}(\Omega).
\end{aligned} \tag{4.18}
$$

Discrete formulation (4.18) is equivalent to searching for the family of vectors $\{\mathbf{u}_K\}_{K\in\mathcal{M}}$ from $\mathbb{R}^2$, and scalars $\{p_K\}_{K\in\mathcal{M}}$ from $\mathbb{R}$ which is the solution to the system of nonlinear equations (SPCFV scheme) obtained by writing the following two equations for each control volume $K \in \mathcal{M}$:

$$
\begin{aligned}
(4.19) \qquad & \eta m_K \mathbf{u}_K - \nu \sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(\mathbf{u}_L - \mathbf{u}_K) + \nu \sum_{\substack{\sigma\in\mathcal{E}_{extD}\\(\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma}\mathbf{u}_K \\
& - \sum_{\substack{\sigma\in\mathcal{E}_{extN}\\(\sigma\in\mathcal{E}_K)}} m_\sigma p_K\, \mathbf{n}_\sigma + \sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}\left[(d_{L,\sigma}\mathbf{u}_K + d_{K,\sigma}\mathbf{u}_L)\cdot \mathbf{n}_{K,\sigma}\right]\mathbf{u}_\sigma \\
& - \left[\lambda \sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(h_K^2 + h_L^2)(p_L - p_K) - \gamma \sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma}\, h_K^2 p_K\right]\mathbf{u}_K \\
& + \sum_{\substack{\sigma\in\mathcal{E}_{extN}\\(\sigma\in\mathcal{E}_K)}} m_\sigma(\mathbf{u}_K\cdot\mathbf{n}_\sigma)\mathbf{u}_K + \sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma} d_{L,\sigma}(p_L - p_K)\mathbf{n}_{K,\sigma} \\
= & \int_K \mathbf{f}\, dx + \sum_{\substack{\sigma\in\mathcal{E}_{extN}\\(\sigma\in\mathcal{E}_K)}} \int_\sigma \mathbf{s}\, d\sigma \\
& \sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(d_{L,\sigma}\mathbf{u}_K + d_{K,\sigma}\mathbf{u}_L)\cdot\mathbf{n}_{K,\sigma} + \sum_{\substack{\sigma\in\mathcal{E}_{extN}\\(\sigma\in\mathcal{E}_K)}} m_\sigma(\mathbf{u}_K\cdot\mathbf{n}_\sigma) \\
& - \lambda \sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(h_K^2 + h_L^2)(p_L - p_K) + \gamma \sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma}\, h_K^2 p_K \\
= & \ 0.
\end{aligned}
$$

In the first equation of the above scheme, the term:

$$
- \left[\lambda \sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(h_K^2 + h_L^2)(p_L - p_K) - \gamma \sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma}\, h_K^2 p_K\right]\mathbf{u}_K
$$

stems from the discrete trilinear form $c_{\mathcal{D}}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ combined with the stabilizing-penalizing term $\langle p, q\rangle_{\mathcal{D},\lambda,\gamma}$.

4.3. Existence and uniqueness of discrete solutions

Because of (4.14), similar arguments as those used in the proof of Lemma 4.1 can be applied to establish that the stability estimates (4.6) and (4.7) also hold for the discrete solutions of problem (4.18). Next, we apply them to prove the existence of these discrete solutions by means of a topological degree argument.

In the sequel, to prove existence of at least one solution to (4.18), the main idea is to apply a so-called topological degree argument in the finite-dimensional case (see, for instance, [7] for the general case). For completeness of presentation, this argument result, first used for numerical schemes in [9], is recalled below.

Theorem 4.2. *Let $\mathbf{V}$ be a finite-dimensional vector space on $\mathbb{R}$ and $g : \mathbf{V} \to \mathbf{V}$ a continuous function. If there exists another continuous function $\mathbf{F} : \mathbf{V} \times [0,1] \to \mathbf{V}$ such that*

(1) *$\mathbf{F}(\cdot, 1) = g$ and $\mathbf{F}(\cdot, 0)$ is an affine function,*
(2) *There is $R > 0$ such that*

$$\forall (\mathbf{v}, \rho) \in \mathbf{V} \times [0,1]\,, \quad \mathbf{F}(\mathbf{v}, \rho) = \mathbf{0} \ \Rightarrow \ \|\mathbf{v}\|_{\mathbf{V}} \neq R,$$

(3) *The equation $\mathbf{F}(\mathbf{v}, 0) = 0$ has a solution $\mathbf{v} \in \mathbf{V}$ with $\|\mathbf{v}\|_{\mathbf{V}} < R$,*

then, the equation $g(\mathbf{v}) = \mathbf{0}$ admits at least a solution $\mathbf{v} \in \mathbf{V}$ with $\|\mathbf{v}\|_{\mathbf{V}} < R$.

In the present case, the equation $g(\mathbf{v}) = \mathbf{0}$ should represent the nonlinear discrete system (4.18), and the task will be to construct the function $\mathbf{F}$ with the required conditions.

Theorem 4.3. *Under all above hypotheses, there exists at least one solution $(\mathbf{u}_h, p_h) \in \mathbf{E}_{\mathcal{D}}(\Omega) \times H_{\mathcal{D}}(\Omega)$ to (4.18). Moreover, if the following uniqueness condition for small data holds*:

$$1 > \frac{2\sqrt{2} N \delta(\Omega)}{\nu^2} \sqrt{\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{s}\|^2_{\mathbf{L}^2(\Gamma_N)}}, \tag{4.20}$$

then this solution $(\mathbf{u}_h, p_h)$ is unique.

Proof. Let us define the finite-dimensional space: $\mathbf{V}_{\mathcal{D}} = \mathbf{E}_{\mathcal{D}}(\Omega) \times H_{\mathcal{D}}(\Omega)$. Consider the mapping $\mathbf{F} : \mathbf{V}_{\mathcal{D}} \times [0,1] \to \mathbf{V}_{\mathcal{D}}$ such that, for given $(\mathbf{u}, p) \in \mathbf{V}_{\mathcal{D}}$ and $\rho \in [0,1]$, $(\hat{\mathbf{u}}, \hat{p}) = \mathbf{F}((\mathbf{u}, p), \rho)$ is defined by:

$$\begin{cases} (\hat{\mathbf{u}}, \mathbf{v})_\Omega = \eta(\mathbf{u}, \mathbf{v})_\Omega + \nu[\mathbf{u}, \mathbf{v}]_{1,\mathcal{D}} + \rho\, c_{\mathcal{D}}(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (p, div_{\mathcal{D}} \mathbf{v})_\Omega \\ \qquad\qquad - (\mathbf{f}, \mathbf{v})_\Omega - (\mathbf{s}, \mathbf{v})_{\Gamma_N} \qquad\qquad \forall \mathbf{v} \in \mathbf{E}_{\mathcal{D}}(\Omega), \\ (\hat{p}, q)_\Omega = (q, div_{\mathcal{D}} \mathbf{u})_\Omega + \langle p, q \rangle_{\mathcal{D},\lambda,\gamma} \qquad\qquad \forall q \in H_{\mathcal{D}}(\Omega). \end{cases} \tag{4.21}$$

Now, let us denote $\hat{\mathbf{u}} = (\hat{u}^{(1)}, \hat{u}^{(2)})$. It is an easy matter to check that the two above relations define a one-to-one function $\mathbf{F}((\cdot,\cdot),\cdot)$. Indeed, for a given $K \in \mathcal{M}$, the values of $\hat{u}^{(1)}_K$, $\hat{u}^{(2)}_K$ and $\hat{p}_K$ are readily obtained by setting in system (4.21): $\mathbf{v} = (1_K, 0)$, $\mathbf{v} = (0, 1_K)$ and $q = 1_K$.

The mapping $\mathbf{F}((\cdot,\cdot),\cdot)$ is clearly a continuous function and, for a given $(\mathbf{u}, p)$ such that $\mathbf{F}((\mathbf{u}, p), \rho) = (\mathbf{0}, 0)$, we can apply estimates (4.6) and (4.7) which prove the boundedness of $(\mathbf{u}, p)$ independently of ρ. Since $\mathbf{F}((\mathbf{u}, p), 0)$ is a bijective affine function of $(\mathbf{u}, p)$ from $\mathbf{E}_{\mathcal{D}}(\Omega)$ into itself (by Theorem 4.1), the existence of at least one solution $(\mathbf{u}_h, p_h) \in \mathbf{E}_{\mathcal{D}}(\Omega) \times H_{\mathcal{D}}(\Omega)$ to the equation $\mathbf{F}((\mathbf{u}, p), 1) = (\mathbf{0}, 0)$, which is exactly (4.18), follows from Theorem 4.2.

Next, we shall prove that the discrete problem has only one solution $(\mathbf{u}_h, p_h)$. In fact, if $(\mathbf{u}_1, p_1)$ and $(\mathbf{u}_2, p_2)$ both satisfy (4.18), then for all $(\mathbf{v}, q) \in \mathbf{E}_{\mathcal{D}}(\Omega) \times$

$H_{\mathcal{D}}(\Omega)$ we would get the compacted identity:

$$\mathcal{B}_h((\mathbf{u}_1-\mathbf{u}_2,p_1-p_2);(\mathbf{v},q))+c_{\mathcal{D}}(\mathbf{u}_1,\mathbf{u}_1,\mathbf{v})-c_{\mathcal{D}}(\mathbf{u}_2,\mathbf{u}_2,\mathbf{v})=0$$

or

$$\mathcal{B}_h((\mathbf{u}_1-\mathbf{u}_2,p_1-p_2);(\mathbf{v},q))+c_{\mathcal{D}}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1,\mathbf{v})+c_{\mathcal{D}}(\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2,\mathbf{v})=0. \tag{4.22}$$

Taking $(\mathbf{v},q)=(\mathbf{u}_1-\mathbf{u}_2,p_1-p_2)$ in (4.22) yields

$$\begin{aligned}&\eta\,\|\mathbf{u}_1-\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)}+\nu\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\mathcal{D}}+\langle p_1-p_2,p_1-p_2\rangle_{\mathcal{D},\lambda,\gamma}\\ =&\;-c_{\mathcal{D}}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1,\mathbf{u}_1-\mathbf{u}_2)-c_{\mathcal{D}}(\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)\end{aligned}$$

so that combining with (4.17) and (4.6) gives

$$\begin{aligned}&\eta\,\|\mathbf{u}_1-\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)}+\nu\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\mathcal{D}}+\min\{\lambda,\gamma\}\,\|p_1-p_2\|^2_{\mathcal{D}}\\ \le&\;|c_{\mathcal{D}}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1,\mathbf{u}_1-\mathbf{u}_2)|+|c_{\mathcal{D}}(\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)|\\ \le&\;N\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\mathcal{D}}\big(\|\mathbf{u}_1\|_{1,\mathcal{D}}+\|\mathbf{u}_2\|_{1,\mathcal{D}}\big)\\ \le&\;\frac{2\sqrt{2}N\delta(\Omega)}{\nu}\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\mathcal{D}}\sqrt{\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}+\|\mathbf{s}\|^2_{\mathbf{L}^2(\Gamma_N)}}.\end{aligned} \tag{4.23}$$

On one hand, we get

$$\nu\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\mathcal{D}}\le\frac{2\sqrt{2}N\delta(\Omega)}{\nu}\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\mathcal{D}}\sqrt{\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}+\|\mathbf{s}\|^2_{\mathbf{L}^2(\Gamma_N)}}$$

which implies

$$\nu\left(1-\frac{2\sqrt{2}N\delta(\Omega)}{\nu^2}\sqrt{\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}+\|\mathbf{s}\|^2_{\mathbf{L}^2(\Gamma_N)}}\right)\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\mathcal{D}}\le 0.$$

The uniqueness condition (4.20) leads to $\mathbf{u}_1=\mathbf{u}_2$. On the other hand, taking into account this and again inequality (4.23) would give

$$\|p_1-p_2\|_{\mathcal{D}}=0$$

and we must have $p_1=p_2$. Consequently, the existence and uniqueness of a discrete solution is established. □

4.4. A convergence analysis of the nonlinear iterative solver

The system of nonlinear equations resulting from the discrete stabilized-penalized problem (4.19) has to solved by means of iterative schemes. For simplicity of presentation, we propose the standard Picard method (see [8]).

Thus, at each Picard iteration, we solve the linearized stabilized-penalized problem which consists in finding $(\mathbf{u}_h^{(k+1)},p_h^{(k+1)})\in\mathbf{E}_{\mathcal{D}}(\Omega)\times H_{\mathcal{D}}(\Omega)$ such that for all $(\mathbf{v},q)\in\mathbf{E}_{\mathcal{D}}(\Omega)\times H_{\mathcal{D}}(\Omega)$:

$$\mathcal{B}_h((\mathbf{u}_h^{(k+1)},p_h^{(k+1)});(\mathbf{v},q))+c_{\mathcal{D}}(\mathbf{u}_h^{(k)},\mathbf{u}_h^{(k+1)},\mathbf{v})=\mathcal{L}(\mathbf{v},q). \tag{4.24}$$

This iteration is clearly well-posed since for given $\mathbf{u}_h^{(k)}$ the linearized version of the trilinear form $c_{\mathcal{D}}$ adds more stability to the bilinear form $\mathcal{B}_h$. Moreover, the estimate (4.6) is also valid for $\mathbf{u}_h^{(k)}$ and $p_h^{(k)}$.

Next, we present a convergence proof of the iteration (4.24).

Theorem 4.4. *Under all hypotheses of Theorem 4.3, the sequence* $(\mathbf{u}_h^{(k)}, p_h^{(k)})$ *given by* (4.24) *converges to* $(\mathbf{u}_h, p_h)$*, solution of* (4.18)*, provided the initial guess* $\mathbf{u}_h^{(0)}$ *is sufficiently close to* $\mathbf{u}_h$ *and the uniqueness condition* (4.20) *holds.*

Proof. We proceed by induction. Set: $\mathbf{e}^{(k)} = \mathbf{u}_h^{(k)} - \mathbf{u}_h$ and $\epsilon^{(k)} = p_h^{(k)} - p_h$. We assume that at an iteration k we have

$$\|\mathbf{e}^{(k)}\|_{1,\mathcal{D}} \leq \chi^k \|\mathbf{e}^{(0)}\|_{1,\mathcal{D}}$$

for some $\chi \in (0,1)$ to be determined, and we seek to show that

$$\|\mathbf{e}^{(k+1)}\|_{1,\mathcal{D}} \leq \chi \|\mathbf{e}^{(k)}\|_{1,\mathcal{D}} \leq \chi^{k+1} \|\mathbf{e}^{(0)}\|_{1,\mathcal{D}}. \tag{4.25}$$

Begin by subtracting the compacted form of (4.18) from (4.24) at the iteration $k{+}1$ to obtain

$$\mathcal{B}_h((\mathbf{e}^{(k+1)}, \epsilon^{(k+1)}); (\mathbf{v}, q)) + c_{\mathcal{D}}(\mathbf{u}_h^{(k)}, \mathbf{u}_h^{(k+1)}, \mathbf{v}) - c_{\mathcal{D}}(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) = 0,$$

or

$$\mathcal{B}_h((\mathbf{e}^{(k+1)}, \epsilon^{(k+1)}); (\mathbf{v}, q)) + c_{\mathcal{D}}(\mathbf{e}^{(k)}, \mathbf{u}_h^{(k+1)}, \mathbf{v}) + c_{\mathcal{D}}(\mathbf{u}_h, \mathbf{e}^{(k+1)}, \mathbf{v}) = 0$$

for all $(\mathbf{v}, q) \in \mathbf{E}_{\mathcal{D}}(\Omega) \times H_{\mathcal{D}}(\Omega)$. Now, let $\mathbf{v} = \mathbf{e}^{(k+1)}$ and $q = \epsilon^{(k+1)}$ in the last equation. By virtue of (4.14), this gives

$$\mathcal{B}_h((\mathbf{e}^{(k+1)}, \epsilon^{(k+1)}); ((\mathbf{e}^{(k+1)}, \epsilon^{(k+1)})) + c_{\mathcal{D}}(\mathbf{e}^{(k)}, \mathbf{u}_h^{(k+1)}, \mathbf{e}^{(k+1)}) = 0.$$

Hence

$$\begin{aligned} \eta \|\mathbf{e}^{(k+1)}\|^2_{\mathbf{L}^2(\Omega)} &+ \nu \|\mathbf{e}^{(k+1)}\|^2_{1,\mathcal{D}} + \min\{\lambda, \gamma\} \|\epsilon^{(k+1)}\|^2_{\mathcal{D}} \\ &\leq |c_{\mathcal{D}}(\mathbf{e}^{(k)}, \mathbf{u}_h^{(k+1)}, \mathbf{e}^{(k+1)})| \\ &\leq N \|\mathbf{e}^{(k)}\|_{1,\mathcal{D}} \|\mathbf{u}_h^{(k+1)}\|_{1,\mathcal{D}} \|\mathbf{e}^{(k+1)}\|_{1,\mathcal{D}} \\ &\leq \frac{N\sqrt{2}\,\delta(\Omega)}{\nu} \sqrt{\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{s}\|^2_{\mathbf{L}^2(\Gamma_N)}} \|\mathbf{e}^{(k)}\|_{1,\mathcal{D}} \|\mathbf{e}^{(k+1)}\|_{1,\mathcal{D}}. \end{aligned} \tag{4.26}$$

If $\|\mathbf{e}^{(k+1)}\|_{1,\mathcal{D}} \neq 0$, this yields

$$\|\mathbf{e}^{(k+1)}\|_{1,\mathcal{D}} \leq \frac{N\sqrt{2}\,\delta(\Omega)}{\nu^2} \sqrt{\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{s}\|^2_{\mathbf{L}^2(\Gamma_N)}} \|\mathbf{e}^{(k)}\|_{1,\mathcal{D}}. \tag{4.27}$$

By invoking (4.20), it is sufficient to select:

$$\chi = \frac{N\sqrt{2}\,\delta(\Omega)}{\nu^2} \sqrt{\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{s}\|^2_{\mathbf{L}^2(\Gamma_N)}} \tag{4.28}$$

to get (4.25), which implies that the velocity iterates $\mathbf{u}_h^{(k)}$ converge to $\mathbf{u}_h$. Another application of (4.26) with (4.27) leads to

$$\sqrt{\lambda}\, \|\epsilon^{(k)}\|_{\mathcal{D}} \leq \chi^k \|\mathbf{e}^{(0)}\|_{1,\mathcal{D}},$$

guaranteeing the convergence of pressure iterates $p_h^{(k)}$ to p_h as well. □

Remark 4.1. 1) The above proof also ensures the convergence of velocity iterates $\mathbf{u}_h^{(k)}$ to $\mathbf{u}_h$ in the $\mathbf{L}^2$-norm.

2) It is important to emphasize that a pressure initial guess $p_h^{(0)}$ is not needed and can be arbitrary.

3) The rate of convergence of Picard iteration is only linear in general, whereas Newton iteration converges quadratically. Nonetheless, the main advantage of Picard iteration, relatively to Newton iteration, comes from the fact that it has a huger ball of convergence. It is reported in [8] that the radius of the convergence ball for Newton iteration is typically proportional to the viscosity parameter ν.

5. Numerical experiments

In this section we assess the computational performance of the SPCFV scheme proposed in the paper. First, we describe the algorithm used for solving the algebraic system (4.19) of nonlinear equations. Then, in order to demonstrate the effectiveness of the numerical SPCFV scheme described and analyzed in the previous sections, we have selected several test generalized steady-state Navier-Stokes problems. The first set of problems with known analytical solutions allows for a study of the convergence rates, whereas the last problem can serve to assess the accuracy and robustness of the SPCFV scheme for different boundary data.

All numerical simulations and figure generations were performed in Matlab on a Intel(R) Core(TM) i5 PC @ 2.53MHz and 4GB of RAM.

5.1. Algorithm

The discrete generalized Navier-Stokes system is solved via the Picard method by choosing the initial guess $(\mathbf{u}_h^{(0)}, p_h^0)$ as the discrete solution of the corresponding linear problem. According to Definition 3.2, we set:

$$\left(\mathbf{u}_K^{(0)}, p_K^{(0)}\right)_{K\in\mathcal{M}} = (\mathbf{u}_h^{(0)}, p_h^{(0)}).$$

By virtue of (4.19), the iterates $\left(\mathbf{u}_K^{(k)}, p_K^{(k)}\right)_{K\in\mathcal{M}} = (\mathbf{u}_h^{(k)}, p_h^{(k)})$ $(k = 1, 2, \ldots)$ are then computed by solving the following system of algebraic linear equations: $(K \in \mathcal{M})$

$$\begin{aligned}(5.1)\qquad & \eta m_K \mathbf{u}_K^{(k)} - \nu \sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(\mathbf{u}_L^{(k)} - \mathbf{u}_K^{(k)}) \\ & + \nu \sum_{\substack{\sigma\in\mathcal{E}_{extD}\\(\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma}\mathbf{u}_K^{(k)} - \sum_{\substack{\sigma\in\mathcal{E}_{extN}\\(\sigma\in\mathcal{E}_K)}} m_\sigma p_K^{(k)}\,\mathbf{n}_\sigma \\ & + \sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}\left[(d_{L,\sigma}\mathbf{u}_K^{(k-1)} + d_{K,\sigma}\mathbf{u}_L^{(k-1)})\cdot\mathbf{n}_{K,\sigma}\right](d_{L,\sigma}\mathbf{u}_K^{(k)} + d_{K,\sigma}\mathbf{u}_L^{(k)})\end{aligned}$$

$$
\begin{aligned}
&- \left[\lambda \sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(h_K^2+h_L^2)(p_L^{(k-1)}-p_K^{(k-1)}) - \gamma \sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma} h_K^2 p_K^{(k-1)} \right] \mathbf{u}_K^{(k)} \\
&+ \sum_{\substack{\sigma\in\mathcal{E}_{extN}\\(\sigma\in\mathcal{E}_K)}} m_\sigma(\mathbf{u}_K^{(k-1)}\cdot\mathbf{n}_\sigma)\,\mathbf{u}_K^{(k)} + \sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma} d_{L,\sigma}(p_L^{(k)}-p_K^{(k)})\,\mathbf{n}_{K,\sigma} \\
= &\int_K \mathbf{f}\,dx + \sum_{\substack{\sigma\in\mathcal{E}_{extN}\\(\sigma\in\mathcal{E}_K)}} \int_\sigma \mathbf{s}\,d\sigma + \sum_{\substack{\sigma\in\mathcal{E}_{extD}\\(\sigma\in\mathcal{E}_K)}} \int_\sigma (\frac{\nu}{d_\sigma} - \mathbf{g}\cdot\mathbf{n}_\sigma)\,\mathbf{g}\,d\sigma \\
&\sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(d_{L,\sigma}\mathbf{u}_K^{(k)} + d_{K,\sigma}\mathbf{u}_L^{(k)})\cdot\mathbf{n}_{K,\sigma} + \sum_{\substack{\sigma\in\mathcal{E}_{extN}\\(\sigma\in\mathcal{E}_K)}} m_\sigma(\mathbf{u}_K^{(k)}\cdot\mathbf{n}_\sigma) \\
&- \lambda \sum_{\substack{L\in\mathcal{N}_K\\(\sigma=K|L)}} \frac{m_\sigma}{d_\sigma}(h_K^2+h_L^2)(p_L^{(k)}-p_K^{(k)}) + \gamma \sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\mathcal{E}_K)}} \frac{m_\sigma}{d_\sigma} h_K^2 p_K^{(k)} \\
= &- \sum_{\substack{\sigma\in\mathcal{E}_{extD}\\(\sigma\in\mathcal{E}_K)}} \int_\sigma \mathbf{g}\cdot\mathbf{n}_\sigma\,d\sigma.
\end{aligned}
$$

We note that, in comparison with (4.19), the right-hand sides in (5.1) contain more terms. This would allow the use of the Dirichlet boundary condition (2.3) with $\mathbf{g} \not\equiv \mathbf{0}$, as it was announced earlier.

The global dimension of the obtained linear system to be solved is equal to $3\text{x}Card(\mathcal{M})$. The outer iterations are terminated once the infinity norm of the difference between two successive iterates falls below a tolerance of 10^{-6}. Finally, the linear solver implemented in Matlab was also used.

5.2. Convergence rates

We first describe numerical experiments to observe the convergence rates of the SPCFV scheme on three generalized steady Navier-Stokes problems of different difficulty and having known analytic solutions. With the domain $\Omega = (0,1)\times(0,1)$, the body force $\mathbf{f}$ given in Ω, the injection velocity $\mathbf{g}$ on $\Gamma_D = \{0\}\times[0,1]\cup[0,1]\times\{0,1\}$ and the traction vector $\mathbf{s}$ on $\Gamma_N = \{1\}\times[0,1]$ are then chosen such that respective exact solutions $(\mathbf{u},p) = (u_1,u_2,p)$ of (2.1)-(2.4) are given in Table 1.

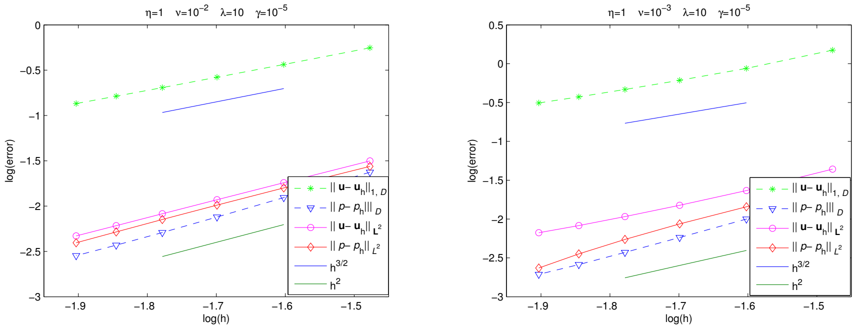
In Test 2, commonly known as Kovasznay flow [19], we choose:

$$\alpha = \frac{1}{2\nu} - \sqrt{\frac{1}{4\nu^2} + 4\pi^2}.$$

For problem discretization, the domain Ω is uniformly subdivided into $n\times n$ square control volumes generating the sequence of meshes SM1, SM2, ..., SM8 with $n = 10i$ ($i = 3,\ldots,8$). There is no satisfactory way to obtain the optimal values for the parameters λ and γ with any given mesh. In practice, these are still being determined by trial and error. More theoretical investigation is still

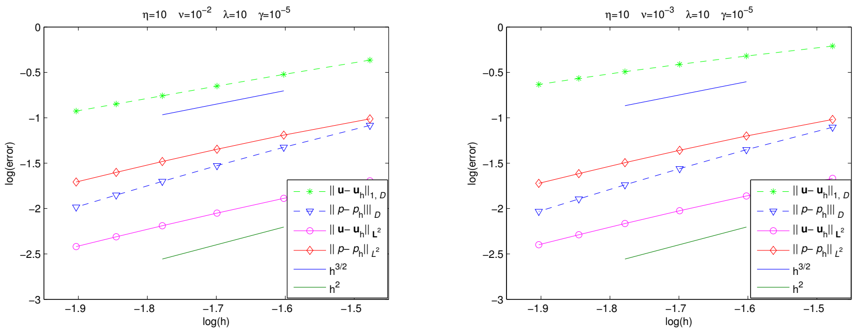
TABLE 1. Exact velocity and pressure solutions for convergence tests.

	$u_1(x,y)$	$u_2(x,y)$	$p(x,y)$
Test 1	$2y(x^2+1)(y-1)(2y-1)$	$-2xy^2(y-1)^2$	xy
Test 2	$1-\mathrm{e}^{\alpha x}\cos 2\pi y$	$\frac{\alpha}{2\pi}\,\mathrm{e}^{\alpha x}\sin 2\pi y$	$\frac{1}{2}(1-\mathrm{e}^{2\alpha x})$
Test 3	0	0	$2x^2(1+x)y(1-y)$



(A) $\nu = 10^{-2}$ (B) $\nu = 10^{-3}$

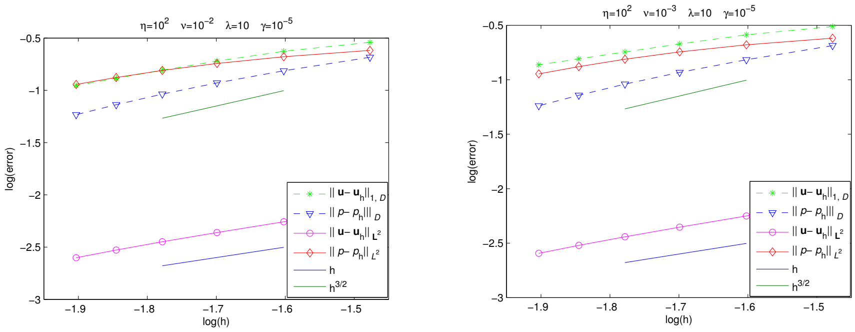
FIGURE 1. Test 1: Convergence history for $(\eta, \lambda, \gamma) = (1, 10, 10^{-5})$.



(A) $\nu = 10^{-2}$ (B) $\nu = 10^{-3}$

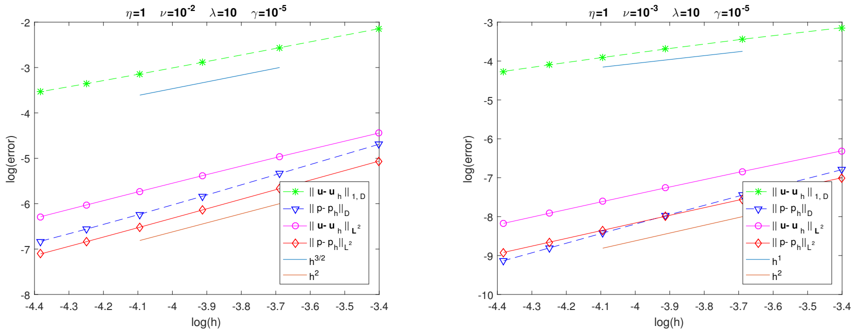
FIGURE 2. Test 1: Convergence history for $(\eta, \lambda, \gamma) = (10, 10, 10^{-5})$.

underway to identify any optimal values. Many fixed values of the stabilization-penalization parameters λ and γ were considered. At least, this suggests that γ must be sufficiently small. However, we present only the representative case



(A) $\nu = 10^{-2}$ (B) $\nu = 10^{-3}$

FIGURE 3. Test 1: Convergence history for $(\eta, \lambda, \gamma) = (10^2, 10, 10^{-5})$.

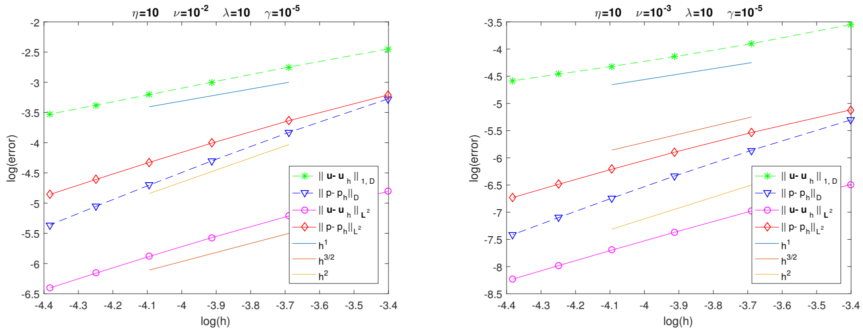


(A) $\nu = 10^{-2}$ (B) $\nu = 10^{-3}$

FIGURE 4. Test 2: Convergence history for $(\eta, \lambda, \gamma) = (1, 10, 10^{-5})$.

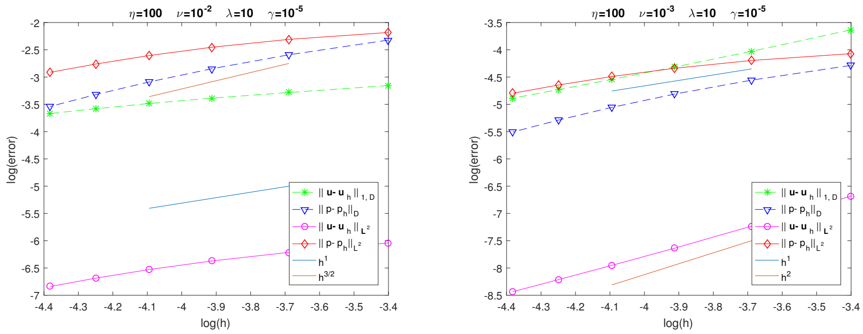
$(\lambda, \gamma) = (10, 10^{-5})$. On the other hand, the kinematic viscosity ν was set to be 10^{-2} and 10^{-3}, whereas the parameter η is considered to be 10^m, $m \in \{0, 1, 2\}$.

We report the convergence rates in different considered norms with respect to the discretization parameter h, which is set to $1/30, 1/40, \ldots, 1/80$. The convergence history in L^2 and discrete H_0^1 norms for both velocity and pressure is depicted in Figures 1-9. An analysis of the displayed results, shows that, for all considered tests and cases, the rates of convergence behave remarkably well. Although appropriate theoretical error estimates are not yet available to us, we may assert that the obtained rates exceed all expectations. Thus, for $\eta = 1$ and $\eta = 10$ the rates show an order of convergence between $\mathcal{O}(h^{3/2})$ and $\mathcal{O}(h^2)$, whereas for $\eta = 10^2$ this order fluctuates between $\mathcal{O}(h)$ and $\mathcal{O}(h^{3/2})$. Moreover, we should emphasize a somehow unexpected behavior of the pressure errors.



(A) $\nu = 10^{-2}$ (B) $\nu = 10^{-3}$

FIGURE 5. Test 2: Convergence history for $(\eta, \lambda, \gamma) = (10, 10, 10^{-5})$.



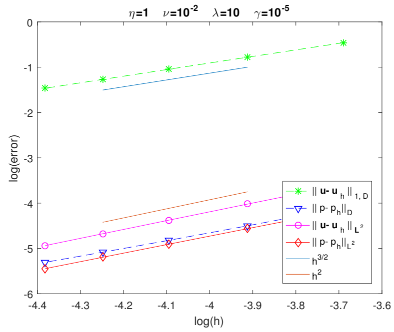
(A) $\nu = 10^{-2}$ (B) $\nu = 10^{-3}$

FIGURE 6. Test 2: Convergence history for $(\eta, \lambda, \gamma) = (10^2, 10, 10^{-5})$.

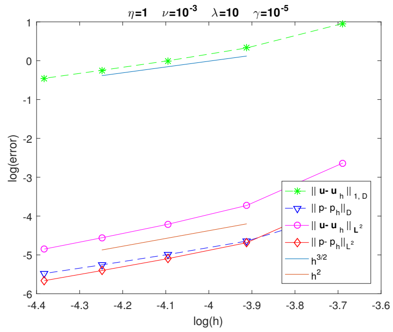
When analyzing for other considered sets of values for all parameters, not reported here, we also noticed that identical results were obtained when the experiments were repeated especially for both L^2 velocity and pressure norms.

5.3. Flow problem in a bifurcated channel

Following our successful experience with usually used backward-facing step problems, not reported here, to examine the robustness and accuracy of the proposed SPCFV scheme, we consider a second generalized Navier-Stokes problem. This is of an incompressible biofluid flow in a domain having three connected components: an inflow channel located at the left part, a transition zone with a facing obstacle in the middle and two outflow subchannels at the two extreme right parts. This problem can play a significant role in validating the efficiency of the scheme. Here, the external body force $\mathbf{f}$ is taken to be zero. Moreover,

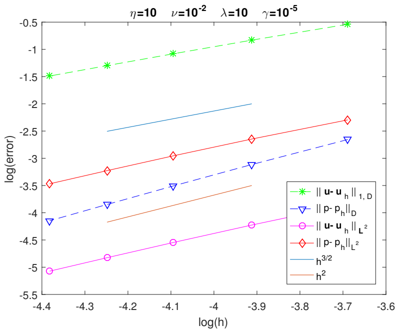


(A) $\nu = 10^{-2}$

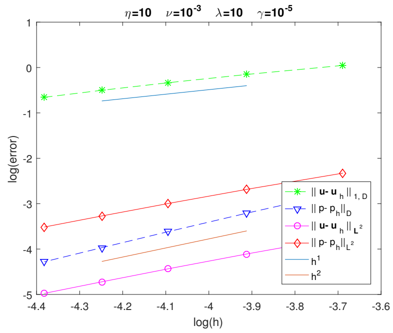


(B) $\nu = 10^{-3}$

FIGURE 7. Test 3: Convergence history for $(\eta, \lambda, \gamma) = (1, 10, 10^{-5})$.



(A) $\nu = 10^{-2}$

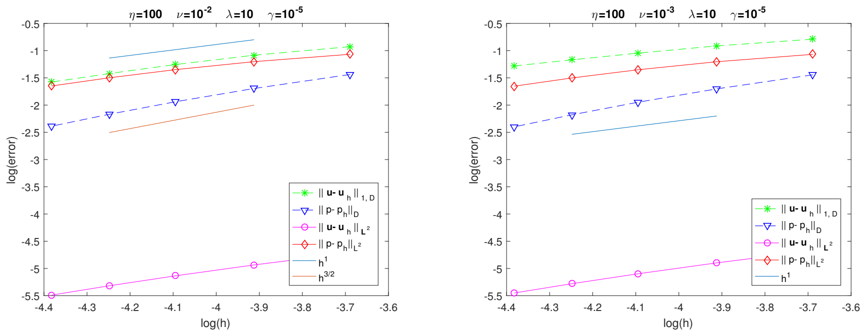


(B) $\nu = 10^{-3}$

FIGURE 8. Test 3: Convergence history for $(\eta, \lambda, \gamma) = (10, 10, 10^{-5})$.

for simplicity of exposition, we shall take the inlet and outlets only in the x-direction. The flow domain Ω with its boundary parts and the coarsest mesh of control volumes are shown in Figure 10. Starting from this mesh of 64 volumes, an usual refining is performed by subdividing every rectangle in Figure 10 into sixty four smaller equal rectangles to generate the computed adaptive mesh of $4,096$ control volumes with $12,288$ degrees of freedom.

Since this problem appears to possess no obvious analytic solution, it is seemingly impossible to calculate the exact accuracy and hence the convergence rates for the discrete velocity and pressure solutions. Fortunately, some features can be exhibited to provide an idea of how the computed solutions behave. To this end, we have used three different numerical tests with the same Dirichlet



(A) $\nu = 10^{-2}$ (B) $\nu = 10^{-3}$

FIGURE 9. Test 3: Convergence history for $(\eta, \lambda, \gamma) = (10^2, 10, 10^{-5})$.

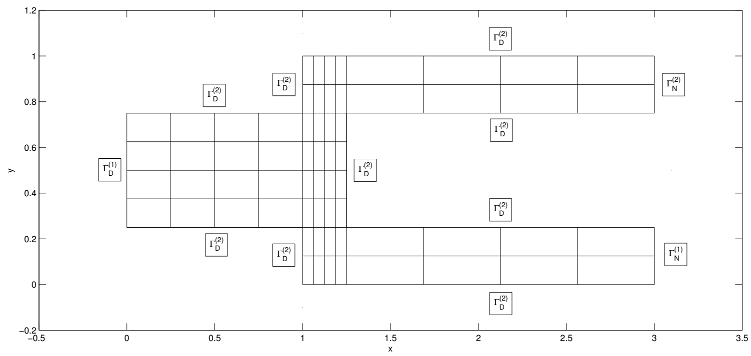


FIGURE 10. Domain Ω with boundary and coarsest mesh.

boundary data on Γ_D:

$$\mathbf{g}\,(x, y) = \begin{cases} [(y - 1/4)(3/4 - y) \quad 0]^T & \text{on } \Gamma_D^{(1)}, \\ \mathbf{0} & \text{on } \Gamma_D^{(2)}, \end{cases} \tag{5.2}$$

whereas the outlet data $\mathbf{s} = [s_1 \; s_2]^T$ on the boundary Γ_N are given according to Table 2.

Although many sets of parameter values were tested, we present the numerical results only for the choice:

$$\eta = 10^2, \qquad \nu = 10^{-4}, \qquad \lambda = 1, \qquad \gamma = 10^{-5}, \tag{5.3}$$

with moderately large η and small viscosity ν.

The results are depicted in Figures 11-19 where the velocity fields and the contours of pressure and velocity components are displayed. In the three cases,

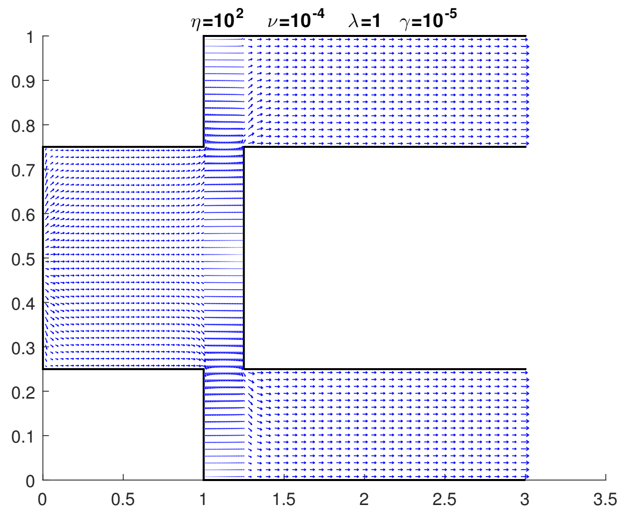


FIGURE 11. Test 1: Velocity field for $\eta = 10^2$ and $\nu = 10^{-4}$.

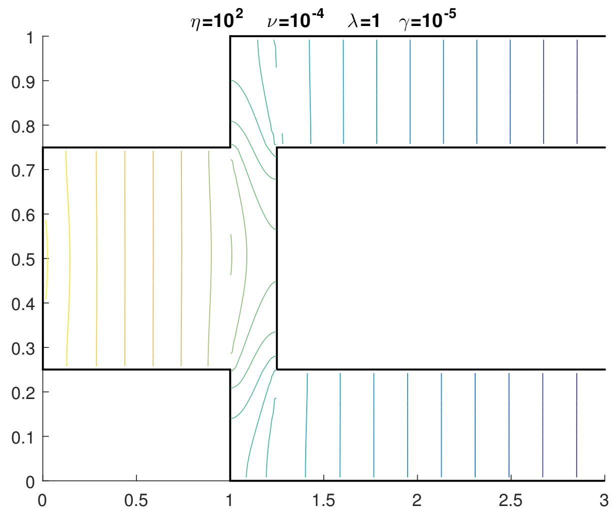
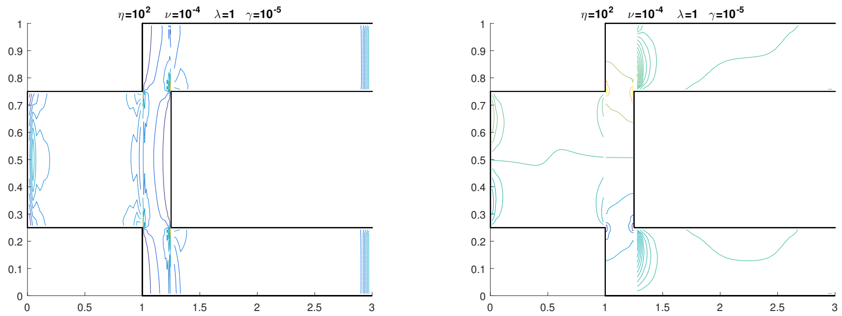


FIGURE 12. Test 1: Pressure contours for $\eta = 10^2$ and $\nu = 10^{-4}$.

the flow behaves as a Poiseuille-type one in the left starting part of the inlet channel and the right extreme parts of the outlet subchannels. Thereby, we observe horizontally directed velocity vectors and equidistantly distributed



(A) $u_h^{(1)}$ (B) $u_h^{(2)}$

Figure 13. Test 1: Contours of velocity components for $\eta = 10^2$ and $\nu = 10^{-4}$.

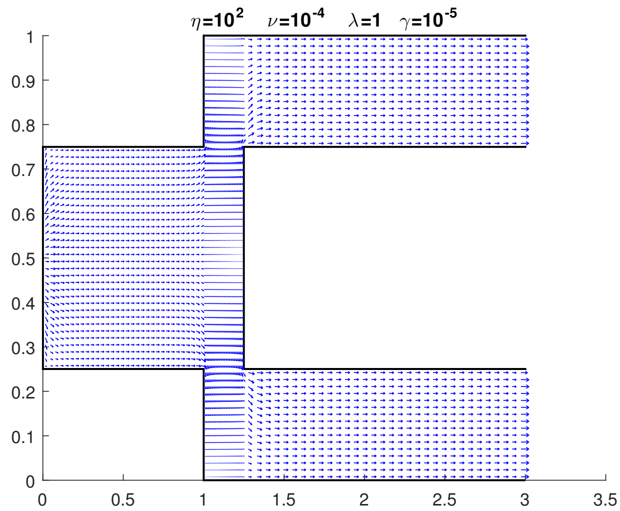


Figure 14. Test 2: Velocity field for $\eta = 10^2$ and $\nu = 10^{-4}$.

vertical pressure isolines. As expected, when the flow approaches the facing obstacle, it seems taking a somehow symmetric bifurcation towards both the lower and upper parts of the transition zone. Accordingly, in this part of the domain, a noticeable recirculation occurs in the transition zone. This is more evidenced by the contour plots of the pressure and the velocity components. It is important to emphasize that all pressure contours are free from any oscillation as it is even more confirmed by Figure 20.

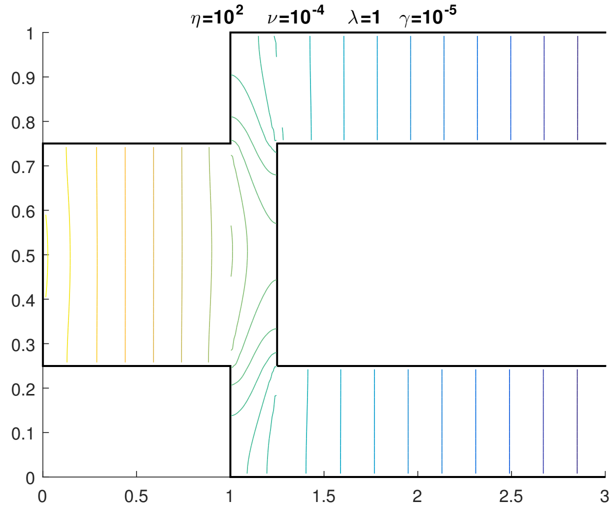
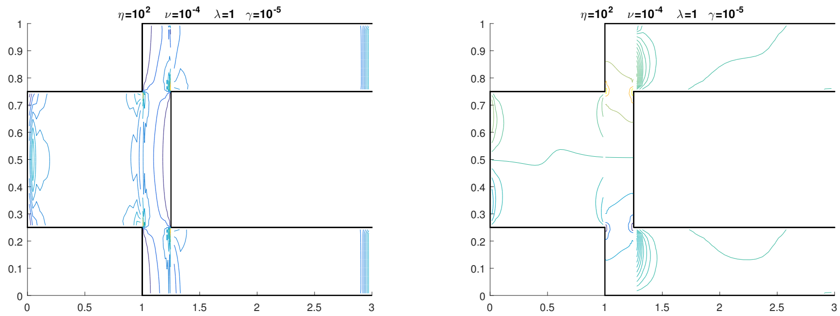


FIGURE 15. Test 2: Pressure contours for $\eta = 10^2$ and $\nu = 10^{-4}$.



(A) $u_h^{(1)}$ (B) $u_h^{(2)}$

FIGURE 16. Test 2: Contours of velocity components for $\eta = 10^2$ and $\nu = 10^{-4}$.

6. Conclusion

In this paper, we have proposed and analyzed a stabilized-penalized collocated finite volume (SPCFV) method for solving the generalized Navier-Stokes equations coupled with mixed Dirichlet-traction boundary conditions. For the discrete solution $(\mathbf{u}_h, p_h)$ we have established the existence, uniqueness and stability. Finally, the reported numerical tests show that the SPCFV scheme

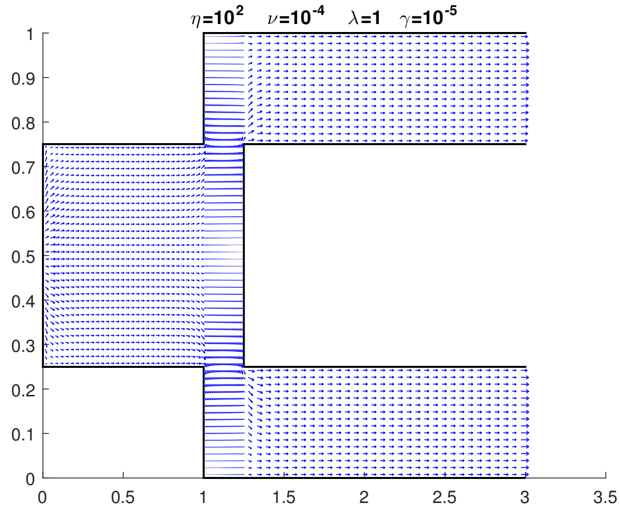


FIGURE 17. Test 3: Velocity field for $\eta = 10^2$ and $\nu = 10^{-4}$.

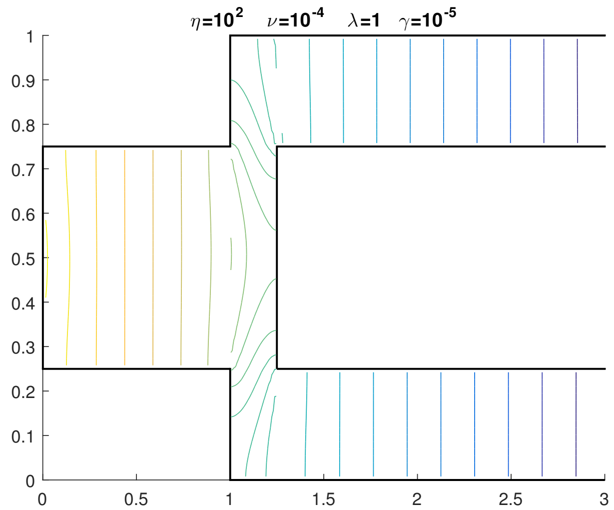
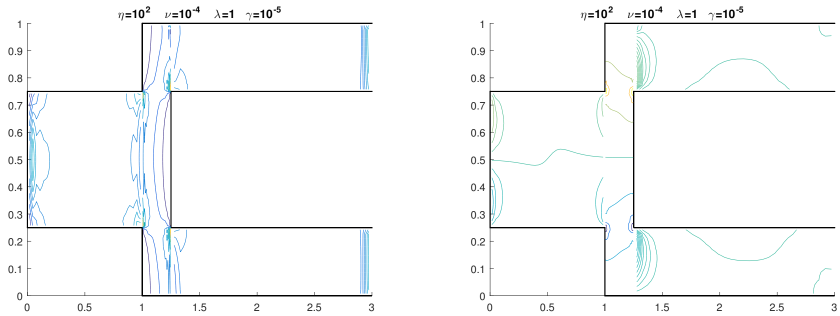


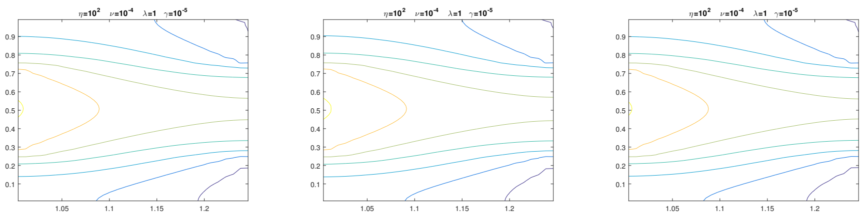
FIGURE 18. Test 3: Pressure contours for $\eta = 10^2$ and $\nu = 10^{-4}$.

can lead to higher convergence rates relatively to the used approximation order. A theoretical rigourous error analysis should be undertaken to answer



(A) $u_h^{(1)}$ (B) $u_h^{(2)}$

FIGURE 19. Test 3: Contours of velocity components for $\eta = 10^2$ and $\nu = 10^{-4}$.



(A) Test 1 (B) Test2 (C) Test 3

FIGURE 20. Pressure contours in the transition zone for $\eta = 10^2$ and $\nu = 10^{-4}$.

TABLE 2. Outlet data for the three tests.

	s_1	s_2
Test 1	0	0
Test 2	1	0
Test 3	$y(1/4-y)$ on $\Gamma_N^{(1)}$ $(y-3/4)(1-y)$ on $\Gamma_N^{(2)}$	0

this issue. The SPCFV scheme is stable and numerically effective for solving the two-dimensional stationary Navier-Stokes equations with moderately small viscosity (or large Reynolds number). Therefore, we hope that it might be suitable to solve practical transient problems arising in biodynamics. We also expect that the performance of the SPCFV scheme can be properly tuned and improved through an informed selection of the stabilization-penalization parameters λ, γ and this has to be reconsidered in a future work.

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