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# ZEROS OF NEW BERGMAN KERNELS

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Abstract. In this paper we determine explicitly the kernels $\mathbb{K}_{\alpha,\beta}$ associated with new Bergman spaces $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D})$ considered recently by the first author and M. Zaway. Then we study the distribution of the zeros of these kernels essentially when $\alpha \in \mathbb{N}$ where the zeros are given by the zeros of a real polynomial $Q_{\alpha,\beta}$. Some numerical results are given throughout the paper.

## 1. Introduction

The notion of Bergman kernels has several applications and represents an essential tool in complex analysis and geometry. This notion was introduced first by Bergman [1], then it has been greatly developed by finding the relationship with other notions as in [6]. Sometimes it is necessary to determine these kernels explicitly. However, this is not simple in general. In fact if an orthonormal basis of a Hilbert space is given, then the Bergman kernel of this space can be obtained as a series using the basis elements. For example, the Bergman kernel of the space $\mathcal{A}^2_\alpha(\mathbb{D})$ of holomorphic functions on the unit disk $\mathbb{D}$ of $\mathbb{C}$ that are square integrable with respect to the positive measure $d\mu_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ is given by $\mathbb{K}_\alpha(z,w) = \frac{1}{(1-z\overline{w})^{\alpha+2}}$. Hence this kernel has no zero in $\mathbb{D}$. For more details about Bergman spaces one can see [4]. In order to obtain kernels with zeros in $\mathbb{D}$, Krantz consider in his book [5] some subspaces of $\mathcal{A}^2_\alpha(\mathbb{D})$. In our statement, instead of considering subspaces, we modify slightly the measure $d\mu_\alpha$ to obtain a Bergman kernel that is comparable in some sense with the previous one with some zeros in $\mathbb{D}$. These spaces are considered recently by N. Ghiloufi and M. Zaway in [3]. We recall the main background of this paper:

Throughout the paper, $\mathbb{D}$ will be the unit disk of the complex plane $\mathbb{C}$ as it was mentioned before and $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. We let $\mathbb{N} := \{0, 1, 2, \dots\}$ be the set of positive integers and $\mathbb{R}$ be the set of real numbers. We use the convention that a real number $x$ is said to be positive (resp. negative) if $x \geq 0$ (resp. $x \leq 0$).

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For every $-1 < \alpha, \beta < +\infty$, we consider the positive measure $\mu_{\alpha,\beta}$ on $\mathbb{D}$ defined by

$$d\mu_{\alpha,\beta}(z) := \frac{1}{\mathscr{B}(\alpha+1,\beta+1)}|z|^{2\beta}(1-|z|^2)^{\alpha}dA(z),$$

where $\mathscr{B}$ is the beta function defined by

$$\mathscr{B}(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}dx = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}, \quad \forall\, s,t > 0$$

and

$$dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta, \quad z = x+iy = re^{i\theta}$$

the normalized area measure on $\mathbb{D}$.

We denote by $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D})$ the set of holomorphic functions on $\mathbb{D}^*$ that belongs to the space:

$$\mathbf{L}^2(\mathbb{D}, d\mu_{\alpha,\beta}) = \{f : \mathbb{D} \to \mathbb{C};\ \text{measurable function such that } \|f\|_{\alpha,\beta,2} < +\infty\},$$

where

$$\|f\|^2_{\alpha,\beta,2} := \int_{\mathbb{D}} |f(z)|^2 d\mu_{\alpha,\beta}(z).$$

The set $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D})$ is a Hilbert space and $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D}) = \mathcal{A}^2_{\alpha,m}(\mathbb{D})$ if $\beta = \beta_0 + m$ with $m \in \mathbb{N}$ and $-1 < \beta_0 \le 0$ (see [3] for more details). We claim here that $\mathcal{A}^2_{\alpha,\beta_0}(\mathbb{D}) = \mathcal{A}^2_{\alpha}(\mathbb{D})$ is the classical Bergman space equipped with the new norm $\|\cdot\|_{\alpha,\beta_0,2}$. Moreover, for any $\alpha, \beta > -1$, if we set

$$e_n(z) = \sqrt{\frac{\mathscr{B}(\alpha+1,\beta+1)}{\mathscr{B}(\alpha+1,n+\beta+1)}}\; z^n \tag{1.1}$$

for every $n \ge -m$, then the sequence $(e_n)_{n\ge -m}$ is an orthonormal basis of $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D})$. Furthermore, if $f, g \in \mathcal{A}^2_{\alpha,\beta}(\mathbb{D})$ with

$$f(z) = \sum_{n=-m}^{+\infty} a_n z^n, \quad g(z) = \sum_{n=-m}^{+\infty} b_n z^n,$$

then

$$\langle f, g\rangle_{\alpha,\beta} = \sum_{n=-m}^{+\infty} a_n \bar{b}_n \frac{\mathscr{B}(\alpha+1,n+\beta+1)}{\mathscr{B}(\alpha+1,\beta+1)},$$

where $\langle\cdot,\cdot\rangle_{\alpha,\beta}$ is the inner product in $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D})$ inherited from $\mathbf{L}^2(\mathbb{D}, d\mu_{\alpha,\beta})$.

The following main result determines the reproducing kernel of $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D})$.

**Theorem 1.1.** *Let $-1 < \alpha, \beta < +\infty$ and $\mathbb{K}_{\alpha,\beta}$ be the reproducing Bergman kernel of $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D})$. Then $\mathbb{K}_{\alpha,\beta}(w,z) = \mathcal{K}_{\alpha,\beta}(w\bar{z})$, where*

$$\mathcal{K}_{\alpha,\beta}(\xi) = \frac{Q_{\alpha,\beta}(\xi)}{(\xi)^m(1-\xi)^{2+\alpha}}$$

*with*

$$Q_{\alpha,\beta}(\xi) = \begin{cases} (\alpha+1)\mathscr{B}(\alpha+1,\beta+1) & \text{if} \quad \beta \in \mathbb{N}, \\ \beta_0 \dfrac{\mathscr{B}(\alpha+1,\beta+1)}{\mathscr{B}(\alpha+1,\beta_0+1)} \displaystyle\sum_{n=0}^{+\infty} \frac{(-\xi)^n}{n+\beta_0} \binom{\alpha+1}{n} & \text{if} \quad \beta \notin \mathbb{N}, \end{cases}$$

*with* $\beta_0 = \beta - \lfloor \beta \rfloor - 1 = \beta - m$.

As a consequence of this main result, the study can be reduced to the case $\beta = \beta_0 \in ]-1,0]$. Indeed if we set

$$\begin{array}{rccc} M: & \mathcal{A}^2_{\alpha,\beta}(\mathbb{D}) & \longrightarrow & \mathcal{A}^2_{\alpha,\beta_0}(\mathbb{D}) \\ & f & \longmapsto & \dfrac{\mathscr{B}(\alpha+1,\beta_0+1)}{\mathscr{B}(\alpha+1,\beta+1)} z^m f, \end{array}$$

then the linear operator $M$ is invertible and bi-continuous and $\mathcal{K}_{\alpha,\beta} = M^{-1} \circ \mathcal{K}_{\alpha,\beta_0}$. Thus we can assume that $m = 0$, i.e., $\beta = \beta_0$.

The proof of the main result is the aim of the following section. Then as a consequence, we will prove that for $\alpha \in \mathbb{N}$ and $\beta \in ]-1,0[$, the zeros set of $\mathbb{K}_{\alpha,\beta}$ is a totally real submanifold of $\mathbb{D}^* \times \mathbb{D}^*$ with real dimension one formed by at most $(\alpha+1)$ connected components. This set is reduced to one connected component for $\beta$ closed to $-1$ ($\beta \to (-1)^+$) and it is empty for $\beta$ near 0 ($\beta \to 0^-$). These zeros are related to the zeros set $\mathcal{Z}_{Q_{\alpha,\beta}}$ of $Q_{\alpha,\beta}$ in $\mathbb{C}$. Hence we will concentrate essentially on the distribution of $\mathcal{Z}_{Q_{\alpha,\beta}}$. This will be the aim of the third section of the paper where we start by a general study and we conclude that $\mathcal{Z}_{Q_{\alpha,\beta}}$ is formed by exactly $(\alpha+1)$ connected regular curves when $\beta$ varies in the interval $]-1,0[$.

We finish the paper with some open problems. Using Python software, some numerical results are investigated in the annex of the paper where we confirm numerically some asymptotic results.

## 2. Proof of the main result

The proof of the first case is simple (it was done in [3]) however, the proof of the second one is more delicate and it will be done by steps. Using the sequence $(e_n)_{n\geq -m}$ given by (1.1), we deduce that the reproducing kernel of $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D})$ can be written as follows:

$$\begin{aligned} \mathbb{K}_{\alpha,\beta}(w,z) &= \sum_{n=-m}^{+\infty} e_n(w)\overline{e_n(z)} = \sum_{n=-m}^{+\infty} \frac{\mathscr{B}(\alpha+1,\beta+1)}{\mathscr{B}(\alpha+1,n+\beta+1)} w^n \overline{z}^n \\ &= \frac{\mathscr{B}(\alpha+1,\beta+1)}{(w\overline{z})^m} \sum_{n=0}^{+\infty} \frac{1}{\mathscr{B}(\alpha+1,n+\beta-m+1)} (w\overline{z})^n \\ &= \frac{\mathcal{R}_{\alpha,\beta}(w\overline{z})}{(w\overline{z})^m} =: \mathcal{K}_{\alpha,\beta}(w\overline{z}), \end{aligned}$$

where

$$\mathcal{R}_{\alpha,\beta}(\xi) = \mathscr{B}(\alpha+1,\beta+1)\sum_{n=0}^{+\infty}\frac{\xi^n}{\mathscr{B}(\alpha+1,n+\beta-m+1)}.$$

This series is well-defined as a consequence of Stirling formula. If $\beta = m \in \mathbb{N}$, then

$$\begin{aligned}\mathcal{R}_{\alpha,m}(\xi) &= \mathscr{B}(\alpha+1,m+1)\sum_{n=0}^{+\infty}\frac{\xi^n}{\mathscr{B}(\alpha+1,n+1)}\\ &= \frac{(\alpha+1)\mathscr{B}(\alpha+1,m+1)}{(1-\xi)^{2+\alpha}}.\end{aligned}$$

We consider now the case $\beta \in ]m-1,m[$ with $m \in \mathbb{N}$ and we prove the result in two steps:

- **First step: The case $\boldsymbol{\alpha} \in \mathbb{N}$.**
  We start by proving the following preliminary lemma.

**Lemma 2.1.** *We have*

$$\mathcal{R}_{\alpha,\beta}(\xi) = \frac{Q_{\alpha,\beta}(\xi)}{(1-\xi)^{2+\alpha}},$$

*where $Q_{\alpha,\beta}$ is a polynomial of degree $\alpha+1$ with $Q_{\alpha,\beta}(1) \neq 0$ that satisfies the recurrence formula*:

$$Q_{\alpha+1,\beta}(\xi) = \frac{1}{\alpha+\beta+2}\left[\xi(1-\xi)Q'_{\alpha,\beta}(\xi)+(\alpha+\beta-m+2+(m-\beta)\xi)Q_{\alpha,\beta}(\xi)\right].$$

*Proof.* If $\alpha = 0$, then we have

$$\begin{aligned}\mathcal{R}_{0,\beta}(\xi) &= \mathscr{B}(1,\beta+1)\sum_{n=0}^{+\infty}\frac{\xi^n}{\mathscr{B}(1,n+\beta-m+1)}\\ &= \frac{1}{\beta+1}\sum_{n=0}^{+\infty}(n+\beta-m+1)\xi^n = \frac{Q_{0,\beta}(\xi)}{(1-\xi)^2}\end{aligned}$$

with

$$Q_{0,\beta}(\xi) = \frac{1}{\beta+1}((m-\beta)\xi+\beta-m+1).$$

Assume that the result is proved for $\alpha \in \mathbb{N}$, i.e.,

$$\mathcal{R}_{\alpha,\beta}(\xi) = \frac{Q_{\alpha,\beta}(\xi)}{(1-\xi)^{2+\alpha}},$$

where $Q_{\alpha,\beta}$ is a polynomial of degree $\alpha+1$ with $Q_{\alpha,\beta}(1) \neq 0$ and we will prove that it is true for $\alpha+1$. Indeed, we have

$$\mathcal{R}_{\alpha+1,\beta}(\xi) = \mathscr{B}(\alpha+2,\beta+1)\sum_{n=0}^{+\infty}\frac{\xi^n}{\mathscr{B}(\alpha+2,n+\beta-m+1)}$$

$$
\begin{aligned}
&= \mathscr{B}(\alpha+2,\beta+1)\sum_{n=0}^{+\infty}\frac{\Gamma(\alpha+3+n+\beta-m)}{\Gamma(\alpha+2)\Gamma(n+\beta-m+1)}\xi^n \\
&= \frac{\mathscr{B}(\alpha+1,\beta+1)}{\alpha+\beta+2}\sum_{n=0}^{+\infty}\frac{(\alpha+2+n+\beta-m)}{\mathscr{B}(\alpha+1,n+\beta-m+1)}\xi^n \\
&= \frac{1}{\alpha+\beta+2}\left(\xi\mathcal{R}'_{\alpha,\beta}(\xi)+(\alpha+\beta-m+2)\mathcal{R}_{\alpha,\beta}(\xi)\right) \\
&= \frac{1}{\alpha+\beta+2}\left(\xi\frac{Q'_{\alpha,\beta}(\xi)}{(1-\xi)^{2+\alpha}}+\xi\frac{(2+\alpha)Q_{\alpha,\beta}(\xi)}{(1-\xi)^{3+\alpha}}\right. \\
&\qquad\qquad \left. +(\alpha+\beta-m+2)\frac{Q_{\alpha,\beta}(\xi)}{(1-\xi)^{2+\alpha}}\right) \\
&= \frac{Q_{\alpha+1,\beta}(\xi)}{(1-\xi)^{3+\alpha}},
\end{aligned}
$$

with

$$
Q_{\alpha+1,\beta}(\xi)=\frac{\xi(1-\xi)Q'_{\alpha,\beta}(\xi)+(\alpha+\beta-m+2+(m-\beta)\xi)Q_{\alpha,\beta}(\xi)}{\alpha+\beta+2}.
$$

Thus $Q_{\alpha+1,\beta}$ is a polynomial of degree $\alpha+2$ and

$$
Q_{\alpha+1,\beta}(1)=\frac{\alpha+2}{\alpha+\beta+2}Q_{\alpha,\beta}(1)\neq 0.
$$

□

*Proof of Theorem 1.1.* Now, we can deduce the proof of Theorem 1.1 in the case $\alpha\in\mathbb{N}$. This will be done by induction on $\alpha$. The result is true for $\alpha=0$. Indeed, we have

$$
Q_{0,\beta}(\xi)=\frac{1}{\beta+1}(1+\beta_0-\beta_0\xi)=\beta_0\frac{\mathscr{B}(1,\beta+1)}{\mathscr{B}(1,\beta_0+1)}\left(\frac{1}{\beta_0}-\frac{\xi}{1+\beta_0}\right).
$$

Assume that the result is true until the value $\alpha$. Thanks to Lemma 2.1, we have

$$
\begin{aligned}
Q_{\alpha+1,\beta}(\xi) &= \frac{1}{\alpha+\beta+2}\left(\xi(1-\xi)Q'_{\alpha,\beta}(\xi)+(\alpha+2+\beta_0-\beta_0\xi)Q_{\alpha,\beta}(\xi)\right) \\
&= \frac{\beta_0\mathscr{B}(\alpha+1,\beta+1)}{(\alpha+\beta+2)\mathscr{B}(\alpha+1,\beta_0+1)}\left[\sum_{j=1}^{\alpha+1} j\frac{(-1)^j}{j+\beta_0}\binom{\alpha+1}{j}\xi^j\right. \\
&\quad + \sum_{j=1}^{\alpha+2}(j-1)\frac{(-1)^{j+1}}{j-1+\beta_0}\binom{\alpha+1}{j-1}\xi^j \\
&\quad + (\alpha+2+\beta_0)\sum_{j=0}^{\alpha+1}\frac{(-1)^j}{j+\beta_0}\binom{\alpha+1}{j}\xi^j
\end{aligned}
$$

$$+\beta_0 \sum_{j=1}^{\alpha+2} \frac{(-1)^j}{j-1+\beta_0} \binom{\alpha+1}{j-1} \xi^j \Bigg]$$

$$= \beta_0 \frac{\mathscr{B}(\alpha+2, \beta+1)}{\mathscr{B}(\alpha+2, \beta_0+1)} \sum_{j=0}^{\alpha+2} \frac{(-\xi)^j}{j+\beta_0} \binom{\alpha+2}{j}.$$

This achieves the first step.

- **Second step: The general case ($\boldsymbol{\alpha > -1}$).**
  We set

$$\begin{aligned} S_{\alpha,\beta_0}(\xi) &:= \frac{\mathscr{B}(\alpha+1, \beta_0+1)}{\beta_0 \mathscr{B}(\alpha+1, \beta+1)} Q_{\alpha,\beta}(\xi) \\ &= \frac{(1-\xi)^{\alpha+2}}{\beta_0} \sum_{n=0}^{+\infty} \frac{\mathscr{B}(\alpha+1, \beta_0+1)}{\mathscr{B}(\alpha+1, n+\beta_0+1)} \xi^n \\ &= \frac{Q_{\alpha,\beta_0}(\xi)}{\beta_0} \end{aligned}$$

and

$$G_{\alpha,\beta_0}(\xi) := \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+\beta_0} \binom{\alpha+1}{n} \xi^n. \tag{2.1}$$

To prove the result it suffices to attest that $S_{\alpha,\beta_0} = G_{\alpha,\beta_0}$ on $\mathbb{D}$. To show this equality we will prove that both functions $S_{\alpha,\beta_0}$ and $G_{\alpha,\beta_0}$ satisfy the following differential equation:

$$\xi F'(\xi) = -\beta_0 F(\xi) + (1-\xi)^{\alpha+1}, \quad \forall\, \xi \in \mathbb{D}. \tag{2.2}$$

It follows that $S_{\alpha,\beta_0} - G_{\alpha,\beta_0}$ satisfies on $\mathbb{D}^*$ the homogenous differential equation: $\xi F'(\xi) = -\beta_0 F(\xi)$. In particular it satisfies the same homogenous differential equation on $]0,1[$. Thus there exists a constant $\sigma \in \mathbb{R}$ such that for every $t \in ]0,1[$ we have $S_{\alpha,\beta_0}(t) - G_{\alpha,\beta_0}(t) = \sigma t^{-\beta_0}$. Since $S_{\alpha,\beta_0} - G_{\alpha,\beta_0}$ is differentiable at 0, we get $\sigma = 0$, i.e., $S_{\alpha,\beta_0} = G_{\alpha,\beta_0}$ on $]0,1[$ and by the analytic extension principle we conclude that $S_{\alpha,\beta_0} = G_{\alpha,\beta_0}$ on $\mathbb{D}$.

To finish the proof we will show that both functions $S_{\alpha,\beta_0}$ and $G_{\alpha,\beta_0}$ satisfy the differential equation (2.2). For $G_{\alpha,\beta_0}$ the result is obvious. Indeed

$$\begin{aligned} \xi G'_{\alpha,\beta_0}(\xi) &= \sum_{n=0}^{+\infty} \frac{n}{n+\beta_0} \binom{\alpha+1}{n} (-\xi)^n \\ &= \sum_{n=0}^{+\infty} \left(1 - \frac{\beta_0}{n+\beta_0}\right) \binom{\alpha+1}{n} (-\xi)^n \\ &= (1-\xi)^{\alpha+1} - \beta_0 G_{\alpha,\beta_0}(\xi). \end{aligned}$$

Now for $S_{\alpha,\beta_0}$, it is not hard to prove that

$$\begin{aligned}\xi S'_{\alpha,\beta_0}(\xi) &= -(1-\xi)^{\alpha+1}\sum_{n=0}^{+\infty}\frac{(\alpha+1)\mathscr{B}(\alpha+1,\beta_0+1)}{(\alpha+1+n+\beta_0)\mathscr{B}(\alpha+1,n+\beta_0+1)}\xi^n\\ &= (1-\xi)^{\alpha+1} - \beta_0 S_{\alpha,\beta_0}(\xi).\end{aligned}$$

Thus the proof of Theorem 1.1 is finished. □

As a first consequence of Theorem 1.1, we obtain the following identity:

**Corollary 2.2.** *Let $-1<\alpha<+\infty$ and $-1<\beta<0$. For every $n\in\mathbb{N}$,*

$$\sum_{k=0}^{n}\binom{\alpha+2}{k}\frac{(-1)^k}{\mathscr{B}(\alpha+1,n-k+\beta+1)} = \frac{\beta}{\mathscr{B}(\alpha+1,\beta+1)}\binom{\alpha+1}{n}\frac{(-1)^n}{n+\beta}.$$

*Proof.* Thanks to Theorem 1.1, we have

$$\begin{aligned}S_{\alpha,\beta}(\xi) &= \sum_{n=0}^{+\infty}\frac{(-1)^n}{n+\beta}\binom{\alpha+1}{n}\xi^n\\ &= \frac{\mathscr{B}(\alpha+1,\beta+1)}{\beta}(1-\xi)^{\alpha+2}\sum_{n=0}^{+\infty}\frac{\xi^n}{\mathscr{B}(\alpha+1,n+\beta+1)}\\ &= \frac{\mathscr{B}(\alpha+1,\beta+1)}{\beta}\left[\sum_{n=0}^{+\infty}\binom{\alpha+2}{n}(-\xi)^n\right]\left[\sum_{n=0}^{+\infty}\frac{\xi^n}{\mathscr{B}(\alpha+1,n+\beta+1)}\right]\\ &= \frac{\mathscr{B}(\alpha+1,\beta+1)}{\beta}\sum_{n=0}^{+\infty}d_n\xi^n,\end{aligned}$$

where

$$d_n = \sum_{k=0}^{n}\binom{\alpha+2}{k}\frac{(-1)^k}{\mathscr{B}(\alpha+1,n-k+\beta+1)}.$$

So the result follows. □

Using the proof of Theorem 1.1, one can conclude the following corollary:

**Corollary 2.3.** *For every $-1<\alpha$ and $-1<\beta<0$, the function $G_{\alpha,\beta}$ defined in* (2.1) *satisfies*:

$$\xi G'_{\alpha,\beta}(\xi) = (1-\xi)^{\alpha+1} - \beta G_{\alpha,\beta}(\xi)$$

*and*

$$\begin{aligned}G_{\alpha+1,\beta}(\xi) &= \frac{1}{\alpha+\beta+2}\left(\xi(1-\xi)G'_{\alpha,\beta}(\xi) + (\alpha+\beta+2-\beta\xi)G_{\alpha,\beta}(\xi)\right)\\ &= \frac{1}{\alpha+\beta+2}\left((\alpha+2)G_{\alpha,\beta}(\xi) + (1-\xi)^{\alpha+2}\right).\end{aligned}$$

*Remarks* 2.4. (1) Using the Stirling formula, one can prove that $G_{\alpha,\beta}$ is bounded on the closed unit disk $\overline{\mathbb{D}}$. This fact will be used frequently in the hole of the paper.

(2) Thanks to Lemma 2.1, for $\alpha \in \mathbb{N}$, one has $G_{\alpha,\beta}(1) \neq 0$. For the general case, if $G_{\alpha_0,\beta}(1) \neq 0$ for some $-1 < \alpha_0 \leq 0$, then $G_{\alpha_0+n,\beta}(1) \neq 0$ for every $n \in \mathbb{N}$.

In the rest of the paper, we assume that $G_{\alpha,\beta}(1) \neq 0$. This may be true for any $-1 < \alpha$ and $-1 < \beta < 0$.

## 3. Zeros of Bergman kernels

Using Theorem 1.1, the function $\mathcal{K}_{\alpha,0}$ has no zero in the unit disk $\mathbb{D}$. However if $-1 < \beta < 0$, then $\mathcal{K}_{\alpha,\beta}$ may have some zeros in $\mathbb{D}$. We claim that if $\xi \in \mathbb{D}^*$ is a zero of $\mathcal{K}_{\alpha,\beta}$, then the sets $\{(z,w) \in \mathbb{D}^2;\ w\overline{z} = \xi\}$ and $\{(z,w) \in \mathbb{D}^2;\ z\overline{w} = \xi\}$ define two totally real algebraic surfaces (of real dimension equal to 2) of $\mathbb{C}^2$ that are contained in the zeros set of the Bergman kernel $\mathbb{K}_{\alpha,\beta}$. Thus it suffices to study the zeros set of $\mathcal{K}_{\alpha,\beta}$.

Due to an algebraic problem, we focus sometimes on the case $\alpha \in \mathbb{N}$, because in this case the zeros of $\mathcal{K}_{\alpha,\beta}$ are given by the zeros of the polynomial $G_{\alpha,\beta}$ contained in $\mathbb{D}$. Thus for $\alpha \in \mathbb{N}$, we will study the zeros set of $G_{\alpha,\beta}$ in the hole complex plane $\mathbb{C}$. It is interesting to discuss the variations of these sets in terms of the parameter $\beta$. All results on $G_{\alpha,\beta}$ can be viewed as particular cases of those of the following linear transformation.

### 3.1. The linear transformation $T_\beta$

If $\mathcal{O}(\mathbb{D}(0,R))$ is the space of holomorphic function on the disk $\mathbb{D}(0,R)$ and $-1 < \beta < 0$, then we define $T_\beta$ on $\mathcal{O}(\mathbb{D}(0,R))$ by

$$T_\beta f(z) = \sum_{n=0}^{+\infty} \frac{a_n}{n+\beta} z^n$$

for any $f(z) = \sum_{n=0}^{+\infty} a_n z^n$. The transformation $T_\beta$ is linear and bijective from $\mathcal{O}(\mathbb{D}(0,R))$ onto itself. It transforms any polynomial to a polynomial with the same degree. We start by the study of zeros of $T_\beta f$ in general then we specialize the study to the case $f(z) = P_\alpha(z) = (1-z)^{\alpha+1}$, where $T_\beta P_\alpha$ is exactly $G_{\alpha,\beta}$.

**Theorem 3.1.** *Let $0 < R \leq +\infty$ and $f$ be a holomorphic function on $\mathbb{D}(0,R)$ such that $(f(0), f'(0)) \neq (0,0)$. Then for every $0 < r_0 < R$, there exist $\beta_1(f,r_0)$ and $\beta_2(f,r_0)$, with*

$$-1 < \beta_1(f,r_0) \leq -\frac{|f(0)|}{|f(0)| + r_0|f'(0)|} \leq \beta_2(f,r_0) < 0,$$

*depending on $f$ and $r_0$ such that the function $T_\beta f$ has no zero in $\mathbb{D}(0,r_0)$ for every $\beta_2(f,r_0) < \beta < 0$ and has exactly one simple zero in $\mathbb{D}(0,r_0)$ for every $-1 < \beta < \beta_1(f,r_0)$.*

When $f(0) = 0$ and $f'(0) \neq 0$ the result is reduced to "0 *is the unique zero* (*simple*) *of the function $T_\beta f$ in $\mathbb{D}(0,r_0)$ for every $-1 < \beta < 0$*". However, when $f'(0) = 0$ and $f(0) \neq 0$ then "*the function $T_\beta f$ has no zero in $\mathbb{D}(0, r_0)$ for every $-1 < \beta < 0$*."

*Proof of Theorem 3.1.* If $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ for every $z \in \mathbb{D}(0, R)$ with $(a_0, a_1) \neq (0, 0)$, then we set

$$F_{\beta,f}(z) = \frac{a_0}{\beta} + \frac{a_1}{1+\beta} z.$$

If $|z| = r_0$ we have

$$|T_\beta f(z) - F_{\beta,f}(z)| \leq \sum_{n=2}^{+\infty} \frac{|a_n|}{n+\beta} r_0^n.$$

Moreover, if we set

$$\psi(\beta) = \left| \frac{|a_0|}{\beta} + \frac{|a_1| r_0}{1+\beta} \right| - \sum_{n=2}^{+\infty} \frac{|a_n|}{n+\beta} r_0^n,$$

then

$$\psi\left( -\frac{|a_0|}{|a_0| + r_0 |a_1|} \right) < 0 \text{ and } \lim_{\beta \to 0^-} \psi(\beta) = +\infty \text{ (resp. } \lim_{\beta \to (-1)^+} \psi(\beta) = +\infty)$$

when $a_0 \neq 0$ (resp. $a_1 \neq 0$). It follows that there exist $\beta_1$ and $\beta_2$, with

$$-1 < \beta_1 \leq -\frac{|a_0|}{|a_0| + r_0 |a_1|} \leq \beta_2 < 0,$$

depending on $f$ and $r_0$ such that for every $\beta \in ]-1, \beta_1[ \cup ]\beta_2, 0[$ one has $\psi(\beta) > 0$. Hence, for every $\beta \in ]-1, \beta_1[ \cup ]\beta_2, 0[$ and $|z| = r_0$, we have $|T_\beta f(z) - F_{\beta,f}(z)| < |F_{\beta,f}(z)|$. Thus by Rouché Theorem, $T_\beta f$ and $F_{\beta,f}$ have the same number of zeros counted with their multiplicities in the disk $\mathbb{D}(0, r_0)$. □

In the following lemma we collect some useful properties of $T_\beta f$ that will be used frequently in the sequel.

**Lemma 3.2.** *If $f$ is a holomorphic function on $\mathbb{D}(0, R)$ and $-1 < \beta < 0$, then the following assertions hold*:

(1) *The number 0 is a zero of $f$ if and only if it is a zero of $T_\beta f$ (with the same multiplicity).*
(2) *The derivative of $T_\beta f$ satisfies*
$$z(T_\beta f)'(z) = f(z) - \beta T_\beta f(z), \quad \forall\, z \in \mathbb{D}(0, R).$$
(3) *The functions $f$ and $T_\beta f$ have a common zero in $\mathbb{D}^*(0, R)$ if and only if the function $T_\beta f$ has a zero in $\mathbb{D}^*(0, R)$ with multiplicity greater than or equal to 2.*

Now we consider a fixed holomorphic function $f$ on $\mathbb{D}(0, R)$ without common zero with $T_\beta$ for any $\beta \in ]-1, 0[$. We set

$$H_f(\beta, z) := T_\beta f(z) = \sum_{n=0}^{+\infty} \frac{a_n}{n+\beta} z^n$$

for $(\beta, z) \in ]-1,0[\times \mathbb{D}(0,R)$ and

$$\mathscr{D}_f := \{(\beta, z) \in ]-1,0[\times \mathbb{D}(0,R);\ H_f(\beta, z) = 0\}.$$

We assume that the set $\mathscr{D}_f$ is not empty. Indeed if $f \equiv c$ is a constant function, then $T_\beta f \equiv \frac{c}{\beta}$, thus $\mathscr{D}_c = \emptyset$ if $c \neq 0$ and $\mathscr{D}_c = ]-1,0[\times \mathbb{C}$ if $c = 0$. Moreover it is easy to find some examples of non constant holomorphic functions $g$ where $T_\beta g$ has no zero for some value of $\beta$. But we don't know if there exists a non constant function $g$ such that $\mathscr{D}_g$ is empty.

**Proposition 3.3.** *The set $\mathscr{D}_f$ is a submanifold of (real) dimension one in $\mathbb{R}^3$ formed by at most countable connected components $(\mathcal{Y}_{f,k})_k$.*

*If $\mathcal{Y}$ is a connected component of $\mathscr{D}_f$, then there exist $-1 \leq a_{\mathcal{Y}} < b_{\mathcal{Y}} \leq 0$ and a $\mathcal{C}^\infty$-function $\mathcal{X} : ]a_{\mathcal{Y}}, b_{\mathcal{Y}}[ \longrightarrow \mathbb{D}(0,R)$ such that*

$$\mathcal{Y} = Graph(\mathcal{X}) := \{(\beta, \mathcal{X}(\beta));\ \beta \in ]a_{\mathcal{Y}}, b_{\mathcal{Y}}[\}.$$

*Moreover for every $\beta \in ]a_{\mathcal{Y}}, b_{\mathcal{Y}}[$, one has*

$$\mathcal{X}'(\beta) = \frac{\mathcal{X}(\beta)}{f(\mathcal{X}(\beta))} \sum_{n=0}^{+\infty} \frac{a_n}{(n+\beta)^2} (\mathcal{X}(\beta))^n. \tag{3.1}$$

*Proof.* For every $(\beta, z) \in \mathscr{D}_f$ we have

$$\frac{\partial H_f}{\partial z}(\beta, z) = (T_\beta f)'(z) = \frac{1}{z}(f(z) - \beta T_\beta(z)) = \frac{1}{z} f(z) \neq 0.$$

The result follows using the implicit functions theorem. □

It is easy to see that if $0 < R < +\infty$, then $a_{\mathcal{Y}} > -1$ for all connected components $\mathcal{Y}$ of $\mathscr{D}_f$ except the unique component $\mathcal{Y}_{f,0}$ given by Theorem 3.1 where $a_{\mathcal{Y}_{f,0}} = -1$. However, $b_{\mathcal{Y}} = 0$ if and only if $R = +\infty$, i.e., $f$ is an entire function. In this case, of entire functions, all functions $\mathcal{X}_{f,k}$ are defined on $]-1,0[$.

*Remark* 3.4. Using the same proof, the previous result can be improved to the complex case as follows: If we set $\Omega := \{\beta \in \mathbb{C};\ -1 < \Re e(\beta) < 0\}$ and $\mathcal{D}_f := \{(\beta, z) \in \Omega \times \mathbb{D}(0,R);\ H_f(\beta, z) = 0\}$, then $\mathcal{D}_f$ is a submanifold of (complex) dimension one in $\Omega \times \mathbb{D}(0,R)$ formed by connected components. Thus, the Lelong number of the current $[\mathcal{D}_f]$ of integration over $\mathcal{D}_f$ is equal to one at every point of $\mathcal{D}_f$. (This is due to the fact that all zeros of $H_f$ are simple.) For more details about currents and Lelong numbers, one can refer to [2].

The following theorem gives the asymptotic behaviors of functions $\mathcal{X}_f$ near $-1$ and $0$ when $f$ is a polynomial. We use the notation $\sim$ to indicate the classical equivalence, i.e., two functions $h_1(t) \underset{t \to t_0}{\sim} h_2(t)$ if we have $\lim_{t \to t_0} \frac{h_1(t)}{h_2(t)} = 1$ whenever $h_2(t) \neq 0$. We claim that if $f(z) = a_0 + a_1 z$, then the solution is explicitly determined by

$$\mathcal{X}_f(\beta) = -\frac{a_0}{a_1} \frac{\beta + 1}{\beta}.$$

Hence we will consider the case when $\deg(f) \geq 2$.

**Theorem 3.5.** *Let $f(z) = \sum_{n=0}^{p} a_n z^n$ be a polynomial of degree $p \geq 2$ with $f(0) \neq 0$. We set $a_n = |a_n| e^{i\theta_n}$ for any $0 \leq n \leq p$. The set $\mathscr{D}_f$ is formed exactly by p connected components $(\mathcal{Y}_{f,k})_{0 \leq k \leq p-1}$ with the corresponding functions $\mathcal{X}_{f,k} : ]-1, 0[ \longrightarrow \mathbb{C}$. Again we keep $\mathcal{X}_{f,0}$ to indicate the function related to the unique component given by Theorem 3.1.*

(1) *For every $0 \leq k \leq p-1$, we have $\lim_{\beta \to 0^-} |\mathcal{X}_{f,k}(\beta)| = +\infty$ and*

$$\mathcal{X}_{f,k}(\beta) \underset{\beta \to 0^-}{\sim} \left( -\frac{p|a_0|}{\beta |a_p|} \right)^{\frac{1}{p}} e^{i \frac{\theta_0 - \theta_p + 2j_k \pi}{p}}, \tag{3.2}$$

*where $j_k \in \mathbb{Z}$ that depends on $k$.*

(2) *If $f'(0) \neq 0$, then for every $1 \leq k \leq p-1$*

$$\lim_{\beta \to (-1)^+} |\mathcal{X}_{f,k}(\beta)| = +\infty \quad \textit{and} \quad \lim_{\beta \to (-1)^+} \mathcal{X}_{f,0}(\beta) = 0.$$

*Moreover, we have*

$$\mathcal{X}_{f,0}(\beta) \underset{\beta \to (-1)^+}{\sim} \frac{a_0}{a_1}(1+\beta) \tag{3.3}$$

*and*

$$\mathcal{X}_{f,k}(\beta) \underset{\beta \to (-1)^+}{\sim} \left( \frac{(p-1)|a_1|}{(1+\beta)|a_p|} \right)^{\frac{1}{p-1}} e^{\frac{i(\theta_1 - \theta_p + (2s_k+1)\pi)}{p-1}}, \quad \forall\, 1 \leq k \leq p-1 \tag{3.4}$$

*for some $s_k \in \mathbb{Z}$ that depends on $k$.*

If $f(0) \neq 0$ and $f'(0) = 0$, then all functions $\mathcal{X}_{f,k}$ are bounded near $-1$.

*Proof.* Let $0 \leq k \leq p-1$. As $a_0 \neq 0$ then using the equality

$$\frac{a_0}{\beta} + \sum_{n=1}^{p} \frac{a_n}{n+\beta} (\mathcal{X}_{f,k}(\beta))^n = 0$$

we obtain

$$\lim_{\beta \to 0^-} \mathcal{X}_{f,k}(\beta) = \infty$$

and

$$-\frac{a_0}{\beta} \underset{\beta \to 0^-}{\sim} \frac{a_p}{p+\beta} (\mathcal{X}_{f,k}(\beta))^p.$$

That means

$$(\mathcal{X}_{f,k}(\beta))^p \underset{\beta \to 0^-}{\sim} -\frac{p a_0}{a_p \beta}$$

so we get Equation (3.2).

With the same way if $a_1 \neq 0$, then for every $0 \leq k \leq p-1$ we have

$$\lim_{\beta \to (-1)^+} \mathcal{X}_{f,k}(\beta) \in \{0, \infty\}.$$

Thanks to Theorem 3.1,

$$\lim_{\beta \to (-1)^+} \mathcal{X}_{f,0}(\beta) = 0 \quad \text{and} \quad \lim_{\beta \to (-1)^+} \mathcal{X}_{f,k}(\beta) = \infty, \quad \forall\, 1 \leq k \leq p-1.$$

Thus, we obtain

$$\mathcal{X}_{f,0}(\beta) \underset{\beta\to(-1)^+}{\sim} -\frac{a_0}{a_1}\frac{1+\beta}{\beta}.$$

Therefore, Equation (3.3) follows.

For $1 \le k \le p-1$ we obtain

$$\frac{a_1}{1+\beta} \underset{\beta\to(-1)^+}{\sim} -\frac{a_p}{p+\beta}(\mathcal{X}_{f,k}(\beta))^{p-1}$$

thus,

$$(\mathcal{X}_{f,k}(\beta))^{p-1} \underset{\beta\to(-1)^+}{\sim} -\frac{(p-1)a_1}{a_p(1+\beta)}$$

and Equation (3.4) follows. □

### 3.2. Application on the Bergman kernels

As mentioned before, for any $\alpha \in \mathbb{N}$, $T_\beta P_\alpha = G_{\alpha,\beta}$, where $P_\alpha(z) = (1-z)^{\alpha+1}$. Hence all previous results are valid and more precisions are needed to accomplish the study of $\mathcal{X}_{\alpha,k} := \mathcal{X}_{P_\alpha,k},\ 0 \le k \le \alpha$. We start by claiming that if $x < 0$, then there exists $\beta_x \in ]-1,0[$ such that $G_{\alpha,\beta_x}(x) = 0$. It follows that $(\beta_x, x)$ is in a component (says $\mathcal{Y}_{\alpha,0}$) of $\mathscr{D}_\alpha := \mathscr{D}_{P_\alpha}$. Hence, the corresponding function $\mathcal{X}_{\alpha,0}$ maps $]-1,0[$ onto $]-\infty,0[$. Indeed we have $\mathcal{X}'_{\alpha,0}(\beta) < 0$ for every $\beta \in ]-1,0[$ thus it is a decreasing function and

$$\lim_{\beta\to 0^-} \mathcal{X}_{\alpha,0}(\beta) = -\infty, \qquad \lim_{\beta\to(-1)^+} \mathcal{X}_{\alpha,0}(r) = 0.$$

Using Corollary 2.3, we can deduce that $\mathcal{X}_{\alpha,0}(\beta) \ge \mathcal{X}_{\alpha+1,0}(\beta)$ for every $\beta \in ]-1,0[$ (See Figure 1).

*Remark* 3.6. For every $\alpha \in \mathbb{N}$, we set $-1 < s_\alpha < 0$ the unique solution of $\mathcal{X}_{\alpha,0}(\beta) = -1$. The polynomial $G_{\alpha,\beta}$ has no zero in $]-1,0[$ for every $s_\alpha < \beta < 0$ and has exactly one simple zero in $]-1,0[$ for every $-1 < \beta < s_\alpha$.

We claim that $(s_\alpha)_\alpha$ is an increasing sequence (See again Figure 1).

The following lemma explain differently the conclusion of Theorem 3.1 in the current statement (See Table 1 for numerical values of $\beta_1$ and $\beta_2$ given by Theorem 3.1 for this example).

**Lemma 3.7.** *For every $\alpha > -1$, the family of functions $(\beta(1+\beta)G_{\alpha,\beta})_{\beta\in]-1,0[}$ converges uniformly on $\mathbb{D}$ to the constant 1 (resp. to the polynomial $(\alpha+1)\xi$) as $\beta \to 0^-$ (resp. as $\beta \to (-1)^+$).*

*In particular, for every $m \in \mathbb{N}$ (resp. $m \in \mathbb{N}^*$) the family of kernels $(\mathcal{K}_{\alpha,\beta})_{\beta\in]m-1,m[}$ converges uniformly on every compact subset of $\mathbb{D}^*$ to $\mathcal{K}_{\alpha,m}$ (resp. to $\mathcal{K}_{\alpha,m-1}$) as $\beta \to m^-$ (resp. as $\beta \to (m-1)^+$).*

*Proof.* The lemma is a simple consequence of the following equality:

$$\beta(1+\beta)G_{\alpha,\beta}(\xi) = (1+\beta) - \beta(1+\alpha)\xi + \beta(1+\beta)\sum_{n=2}^{+\infty}\frac{(-\xi)^n}{n+\beta}\binom{\alpha+1}{n}$$

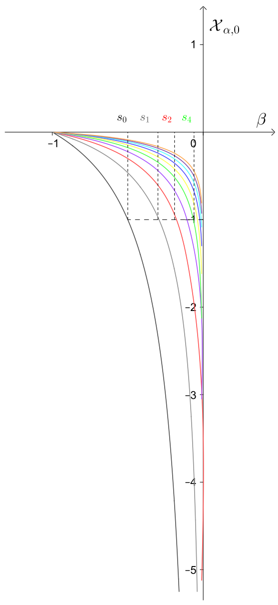


FIGURE 1. Graphs of $\mathcal{X}_{\alpha,0}$ for $0 \leq \alpha \leq 9$.

and the fact that the series converges normally on $\mathbb{D}$ (obtained using the Stirling formula). $\square$

TABLE 1. Numerical values of $\beta_1(P_\alpha, 1)$ and $\beta_2(P_\alpha, 1)$ given by Theorem 3.1.

| $\alpha$ | $\beta_1(P_\alpha, 1)$ | $\beta_2(P_\alpha, 1)$ |
|---|---|---|
| 2 | $-0.381966$ | $-0.177124$ |
| 3 | $-0.493058$ | $-0.107610$ |
| 4 | $-0.667086$ | $-0.0649539$ |
| 5 | $-0.793482$ | $-0.0387481$ |
| 6 | $-0.870294$ | $-0.0227925$ |
| 7 | $-0.917737$ | $-0.0132128$ |
| 8 | $-0.947843$ | $-0.00755239$ |
| 9 | $-0.967185$ | $-0.0042614$ |

The most important conclusion of this lemma is the continuity of the Bergman kernels $\mathbb{K}_{\alpha,\beta}$ in terms of the parameter $\beta$. Essentially the fact that the Bergman kernel $\mathbb{K}_{\alpha,\beta}$ converges uniformly on every compact subset of $\mathbb{D}^2$ to the classical Bergman kernel $\mathbb{K}_{\alpha,0}$ when $\beta \to 0^-$.

Now we will focus on the other components of $\mathscr{D}_\alpha$. We use $\mathcal{X}_{\alpha,k},\ 0 \leq k \leq \alpha$ to indicate the corresponding functions such that $\Im m(\mathcal{X}_{\alpha,k}(\beta)) \leq 0$ for every

$0 \le k \le \lfloor \frac{\alpha+1}{2} \rfloor$ and $\mathcal{X}_{\alpha,\alpha+1-k}(\beta) = \overline{\mathcal{X}}_{\alpha,k}(\beta)$ for every $1 \le k \le \alpha$. Theorem 3.5 can be written as follows:

**Proposition 3.8.** *For every $\alpha \in \mathbb{N}$, we have*

$$\begin{cases} \mathcal{X}_{\alpha,k}(\beta) & \underset{\beta\to 0^-}{\sim} \left(-\dfrac{\alpha+1}{\beta}\right)^{\frac{1}{\alpha+1}} e^{i\frac{(2k-\alpha-1)\pi}{\alpha+1}}, \quad \forall\, 0 \le k \le \alpha, \\ \mathcal{X}_{\alpha,k}(\beta) & \underset{\beta\to(-1)^+}{\sim} \left(\dfrac{\alpha(\alpha+1)}{1+\beta}\right)^{\frac{1}{\alpha}} e^{\frac{i(2k-\alpha-1)\pi}{\alpha}}, \quad \forall\, 1 \le k \le \alpha, \\ \mathcal{X}_{\alpha,0}(\beta) & \underset{\beta\to(-1)^+}{\sim} -\dfrac{1+\beta}{\alpha+1}. \end{cases}$$

The following figures (Figures 2 and 3) explain numerically the result of Proposition 3.8.

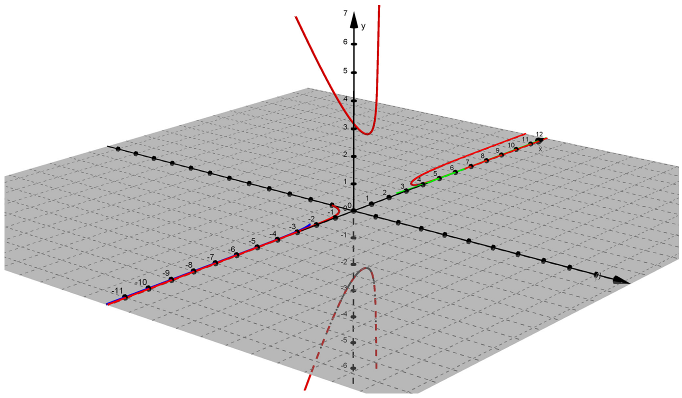

FIGURE 2. Graphs of $\mathcal{X}_{3,\bullet}$ (in red) with asymptotic curves (to $\mathscr{C}_{\mathcal{X}_{3,0}}$ in blue and to $\mathscr{C}_{\mathcal{X}_{3,2}}$ in green)

### 3.3. Even and odd Bergman kernels

Following the idea of Krantz developed in [5], we consider the subspaces $\mathscr{E}^2_{\alpha,\beta}(\mathbb{D})$ and $\mathscr{L}^2_{\alpha,\beta}(\mathbb{D})$ of $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D})$ generated respectively by the even $(e_{2n})_{n\ge 0}$ and the odd $(e_{2n+1})_{n\ge 0}$ sequences. Hence $\mathscr{E}^2_{\alpha,\beta}(\mathbb{D})$ and $\mathscr{L}^2_{\alpha,\beta}(\mathbb{D})$ are Hilbert subspaces of $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D})$ formed respectively by even and odd functions. The reproducing Bergman kernels of these spaces are given by $\mathbb{E}_{\alpha,\beta}(z,w) = \mathcal{E}_{\alpha,\beta}(z\overline{w})$ and $\mathbb{L}_{\alpha,\beta}(z,w) = \mathcal{L}_{\alpha,\beta}(z\overline{w})$, where

$$\begin{aligned} \mathcal{E}_{\alpha,\beta}(\xi) &= \frac{1}{2}(\mathcal{K}_{\alpha,\beta}(\xi) + \mathcal{K}_{\alpha,\beta}(-\xi)) \\ &= \frac{1}{2(1-\xi^2)^{\alpha+2}} \big((1+\xi)^{\alpha+2} Q_{\alpha,\beta}(\xi) + (1-\xi)^{\alpha+2} Q_{\alpha,\beta}(-\xi)\big) \end{aligned}$$

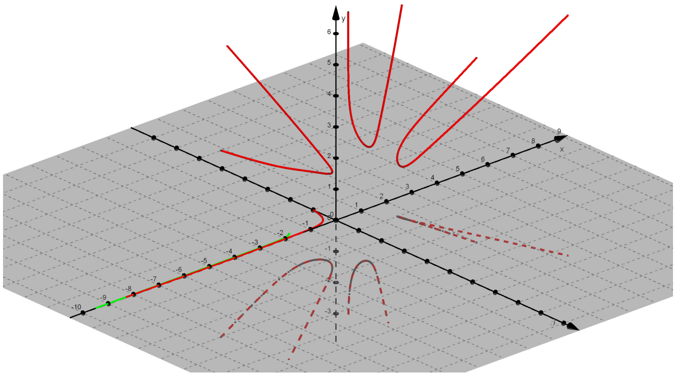

FIGURE 3. Graphs of $\mathcal{X}_{6,\bullet}$ (in red) with asymptotic curve to $\mathscr{C}_{\mathcal{X}_{6,0}}$ (in green)

$$=: \frac{\mathcal{I}_{\alpha,\beta}(\xi)}{2(1-\xi^2)^{\alpha+2}}$$

and

$$\begin{aligned}\mathcal{L}_{\alpha,\beta}(\xi) &= \frac{1}{2}(\mathcal{K}_{\alpha,\beta}(\xi) - \mathcal{K}_{\alpha,\beta}(-\xi)) \\ &= \frac{1}{2(1-\xi^2)^{\alpha+2}}\left((1+\xi)^{\alpha+2}Q_{\alpha,\beta}(\xi) - (1-\xi)^{\alpha+2}Q_{\alpha,\beta}(-\xi)\right) \\ &=: \frac{\mathcal{J}_{\alpha,\beta}(\xi)}{2(1-\xi^2)^{\alpha+2}}.\end{aligned}$$

Again, to study the zeros of even and odd Bergman kernels, it suffices to study the zeros of the corresponding functions $\mathcal{I}_{\alpha,\beta}$ and $\mathcal{J}_{\alpha,\beta}$. Let $\varepsilon_{\alpha,\beta}$ (resp. $\Theta_{\alpha,\beta}$) be the number of zeros of the function $\mathcal{I}_{\alpha,\beta}$ (resp. $\mathcal{J}_{\alpha,\beta}$) in the unit disk $\mathbb{D}$ counted with their multiplicities. To determine $\varepsilon_{\alpha,\beta}$ and $\Theta_{\alpha,\beta}$ in the case when $\alpha \in \mathbb{N}$, we start by the case $\beta = 0$. In this case it is easy to check that the zeros of $\mathcal{I}_{\alpha,0}$ are given by $z_k := -i\tan\left(\frac{(2k+1)\pi}{2(\alpha+2)}\right)$ where $0 \le k \le \alpha+1$ (we omit the value $k$ for which $\cos\left(\frac{(2k+1)\pi}{2(\alpha+2)}\right) = 0$ whenever $\alpha$ is odd). Similarly to the even case, the zeros of $\mathcal{J}_{\alpha,0}$ are given by $w_k := -i\tan\left(\frac{k\pi}{\alpha+2}\right)$, $0 \le k \le \alpha+1$. It follows that if $\alpha = 4\tau + r$ with $\tau \in \mathbb{N}$ and $0 \le r \le 3$, then

$$\varepsilon_{\alpha,0} = \begin{cases} 2\tau & \text{if } \ r = 0, \\ 2\tau+2 & \text{if } \ 1 \le r \le 3, \end{cases} \qquad \Theta_{\alpha,0} = \begin{cases} 2\tau+1 & \text{if } \ 0 \le r \le 2, \\ 2\tau+3 & \text{if } \ r = 3. \end{cases}$$

**Proposition 3.9.** *Let $\alpha = 4\tau + r \in \mathbb{N}$ with $\tau \in \mathbb{N}$ and $0 \le r \le 3$.*

(1) *If $r \neq 0$, then*

 (a) *There exists $-1 < \beta_4 < 0$ such that for every $\beta_4 < \beta \le 0$ we have $\varepsilon_{\alpha,\beta} = \varepsilon_{\alpha,0}$.*

(b) *There exists* $-1 < \beta_5 < 0$ *such that for every* $-1 < \beta < \beta_5$ *we have* $\Theta_{\alpha,\beta} = \varepsilon_{\alpha,0} + 1$.

(2) *If* $r \neq 2$, *then*
  (a) *There exists* $-1 < \beta_3 < 0$ *such that for every* $-1 < \beta < \beta_3$ *we have* $\varepsilon_{\alpha,\beta} = \Theta_{\alpha,0} + 1$.
  (b) *There exists* $-1 < \beta_6 < 0$ *such that for every* $\beta_6 < \beta \leq 0$ *we have* $\Theta_{\alpha,\beta} = \Theta_{\alpha,0}$.

*Proof.* We claim that $\pm i$ are zeros of $\mathcal{I}_{\alpha,0}$ (resp. $\mathcal{J}_{\alpha,0}$) when $\alpha = 4\tau$ (resp. $\alpha = 4\tau + 2$). For this reason we omit the corresponding values of $\alpha$ in the proposition in order to use the Rouché theorem. Hence it suffices to study the convergence in terms of the parameter $\beta$.

Thanks to Lemma 3.7, the family of polynomials $((1 + \beta)Q_{\alpha,\beta}(\xi))_{-1<\beta<0}$ converges to 1 on $\mathbb{D}$ when $\beta \to 0^-$ and to the polynomial $(\alpha + 1)\xi$ when $\beta \to (-1)^+$. It follows that $(1+\beta)\mathcal{I}_{\alpha,\beta}(\xi)$ converges to $\mathcal{I}_{\alpha,0}(\xi)$ on $\mathbb{D}$ as $\beta \to 0^-$ and to $(\alpha+1)\xi\mathcal{J}_{\alpha,0}(\xi)$ on $\mathbb{D}$ as $\beta \to (-1)^+$.

For the odd case, the family $(1+\beta)\mathcal{J}_{\alpha,\beta}(\xi)$ converges to $\mathcal{J}_{\alpha,0}(\xi)$ when $\beta \to 0^-$ and to $(\alpha+1)\xi\mathcal{I}_{\alpha,0}(\xi)$ when $\beta \to (-1)^+$. Using the Rouché theorem the result follows. □

To improve the previous result, we consider the number of zeros $\widehat{\varepsilon}_{\alpha,0}$ (resp. $\widehat{\Theta}_{\alpha,0}$) of the function $\mathcal{I}_{\alpha,0}$ (resp. $\mathcal{J}_{\alpha,0}$) in the closed unit disk $\overline{\mathbb{D}}$ given by $\widehat{\varepsilon}_{\alpha,0} = 2\tau + 2$ and

$$\widehat{\Theta}_{\alpha,0} = \begin{cases} 2\tau + 1 & \text{if} \quad 0 \leq r \leq 1, \\ 2\tau + 3 & \text{if} \quad 2 \leq r \leq 3, \end{cases}$$

where $\alpha = 4\tau + r$. Using the same idea, one can prove the following corollary:

**Corollary 3.10.** *Let* $\alpha \in \mathbb{N}$ *and* $\eta_0 = \tan\left(\frac{\pi}{4} + \frac{\pi}{\alpha+2}\right)$.

(1) *There exist* $-1 < \beta_3 < \beta_4 < 0$ *that depend on* $\alpha$ *such that for every* $1 < \eta < \eta_0$, *the polynomial* $\mathcal{I}_{\alpha,\beta}(\eta\xi)$ *has exactly* $\widehat{\varepsilon}_{\alpha,0}$ *zeros in* $\mathbb{D}$ *for every* $\beta \in ]\beta_4, 0]$ *and* $\widehat{\Theta}_{\alpha,0} + 1$ *zeros in* $\mathbb{D}$ *for every* $\beta \in ] - 1, \beta_3[$.
(2) *There exist* $-1 < \beta_5 < \beta_6 < 0$ *that depend on* $\alpha$ *such that for every* $1 < \eta < \eta_0$, *the polynomial* $\mathcal{J}_{\alpha,\beta}(\eta\xi)$ *has exactly* $\widehat{\Theta}_{\alpha,0}$ *zeros in* $\mathbb{D}$ *for every* $\beta \in ]\beta_6, 0]$ *and* $\widehat{\varepsilon}_{\alpha,0} + 1$ *zeros in* $\mathbb{D}$ *for every* $\beta \in ] - 1, \beta_5[$.

If the conditions of the previous proposition are satisfied, then one can take $\eta = 1$ in the corollary to obtain the same result given by the proposition.

## 4. Open problems

It is interesting to study the asymptotic distribution of zeros of $G_{\alpha,\beta}$ when $\alpha \in \mathbb{N}$ and goes to infinity. In other words, can we find a positive measure $\mu$ such that the sequence of measures

$$\mu_{\alpha,\beta} := \frac{1}{\alpha + 1} \sum_{j=0}^{\alpha+1} \delta_{\mathcal{X}_{\alpha,j}(\beta)}$$

converges weakly to the measure $\mu$ as $\alpha \to +\infty$? Geometrically, the distribution of the set $\{\mathcal{X}_{\alpha,j}(\beta),\ 0 \le j \le \alpha+1\}$ may depend on $\alpha$ in some non trivial way. For example, the distribution of sets $\mathcal{X}_{\alpha,\bullet}(-10^{-4})$ for $\alpha = 31, 32, 33, 34, 36$ are similar (see Figure 4) however these are different to the one that correspond to $\alpha = 35$ (see Figure 5).

Can we find explicitly the equation of the parametric curve that describe the set $\mathcal{X}_{\alpha,\bullet}(\beta)$? (This curve may be a circle in Figure 4 for $\alpha = 31, 32, 33, 34, 36$.) See also Figures 5 and 6.

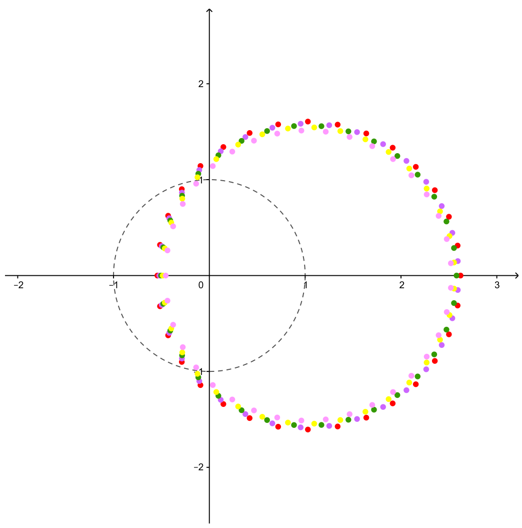


FIGURE 4. The sets $\mathcal{X}_{\alpha,\bullet}(-10^{-4})$ for $\alpha \in \{31, 32, 33, 34, 36\}$

It is simple to prove that for every $1 \le k \le \alpha$, there exists $t_{\alpha,k} \in ]-1, 0[$ such that

$$|\mathcal{X}_{\alpha,k}(t_{\alpha,k})| = \min_{-1<\beta<0} |\mathcal{X}_{\alpha,k}(\beta)|$$

and satisfies

$$\sum_{j,k=0}^{\alpha+1} \binom{\alpha+1}{j}\binom{\alpha+1}{k} \frac{(-1)^{j+k}}{(j+t_{\alpha,k})^2} R_{\alpha,k}^{j+k} \cos(\theta_{\alpha,k}(j-k)) = 0$$

with $\mathcal{X}_{\alpha,k}(t_{\alpha,k}) = R_{\alpha,k} e^{i\theta_{\alpha,k}}$.

One of the most important questions is to see if the critical value $t_{\alpha,k}$ of $\beta$ that realizes the minimum of $|\mathcal{X}_{\alpha,k}(\beta)|$ doesn't depend on $k$. It means that all functions attempt their minimums at the same "time".

For even and odd kernels, can we prove Corollary 3.10 with $\eta = 1$? Indeed, if we show that the zeros of $\mathcal{I}_{\alpha,\beta}(\xi)$ and $\mathcal{J}_{\alpha,\beta}(\xi)$ that converges to $\pm i$ are in $\mathbb{D}$, then we conclude the result. We note that this fact is confirmed numerically for some values of $\alpha$.

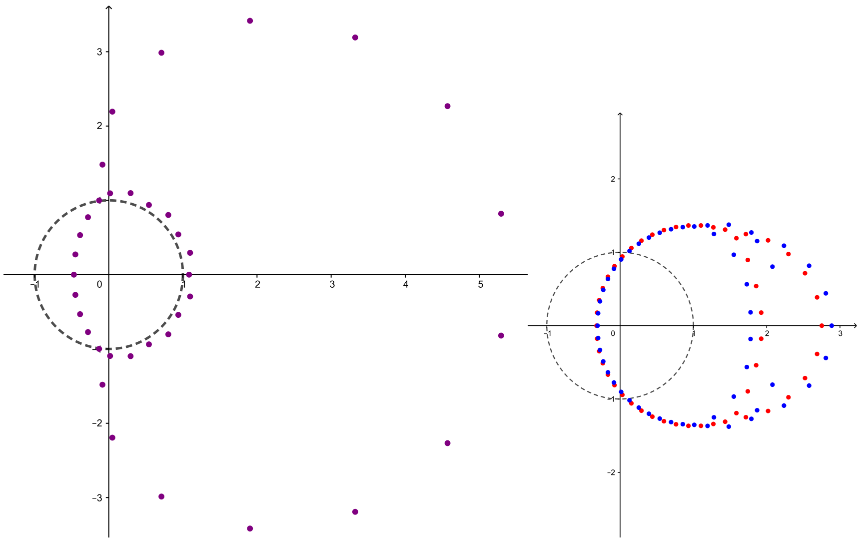


FIGURE 5. The sets $\mathcal{X}_{\alpha,\bullet}(-10^{-4})$ for $\alpha = 35$ (in violet) at left and $\alpha = 49$ (in red) and $\alpha = 51$ (in blue) at right

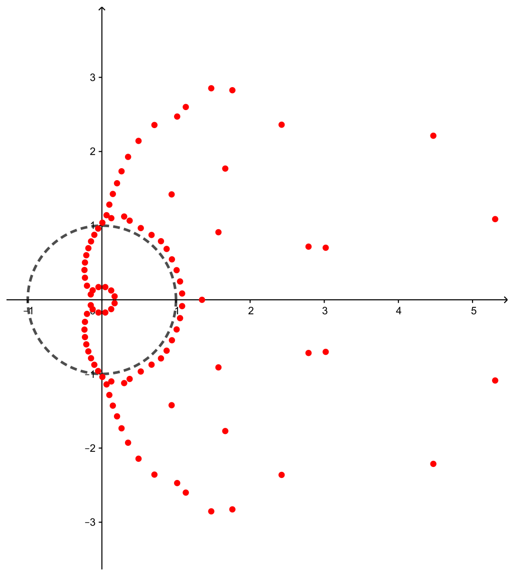


FIGURE 6. The set $\mathcal{X}_{101,\bullet}(-10^{-4})$

## Annex: Numerical results

All figures of this paper were produced using Python software. We give here the used code.

*********************************************

```
from scipy import special
from sympy.abc import x, y, z
def A(beta, alpha):
 s=0
 for j in range(0, alpha+2):
 s=s+ ((1)/(j+beta)*special.binom(alpha+1,j)*(-x)**j)
 return s
import numpy
from cmath import *
from sympy.solvers import solve
import csv
DATA_PATH = '/content/drive/My Drive/graphes data/alpha7_7.csv'
i=1
with open(DATA_PATH, mode='w', newline='') as points_file:
 points_writer = csv.writer(points_file, delimiter=',')
 for beta in numpy.arange(10**(-6), 1, 10**(-2)):
 row = []
 for s in solve(A(beta, 6), x):
 sol = complex(s)
 row.append(-beta)
 row.append(sol.real)
 row.append(sol.imag)
 points_writer.writerow(row)
 print(i)
 i=i+1
 i=1
 for beta in numpy.arange(1-10**(-2), 1, 10**(-3)):
 row = []
 for s in solve(A(beta, 6), x):
 sol = complex(s)
 row.append(-beta)
 row.append(sol.real)
 row.append(sol.imag)
 points_writer.writerow(row)
 print(i)
 i=i+1
```

*********************************************

In Figures 4, 5, 6, we present some zeros sets of $G_{\alpha,\beta}$ for $\beta = -10^{-4}$ and $\alpha \in \{31, 32, 33, 34, 35, 36, 49, 51, 101\}$. The values of $\alpha$ and $\beta$ are chosen arbitrary just to see that there is no geometric stability of these zeros. It is possible that the geometric distribution of zeros of $G_{\alpha,\beta}$ depends on both $\alpha$ and $\beta$ in a complicate manner. It is also possible that if we see numerically the geometric distribution of zeros of $G_{\alpha,\beta}$ for $\alpha$ large enough, then some new limit curve appear. However as a material problem, it was not possible for us to exceed the value $\alpha = 101$.

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