BLOW-UP OF SOLUTIONS FOR WAVE EQUATIONS WITH STRONG DAMPING AND VARIABLE-EXPONENT NONLINEARITY

Sun-Hye Park

ABSTRACT. In this paper we consider the following strongly damped wave equation with variable-exponent nonlinearity

$$u_{tt}(x,t) - \Delta u(x,t) - \Delta u_t(x,t) = |u(x,t)|^{p(x)-2} u(x,t),$$

where the exponent $p(\cdot)$ of nonlinearity is a given measurable function. We establish finite time blow-up results for the solutions with non-positive initial energy and for certain solutions with positive initial energy. We extend the previous results for strongly damped wave equations with constant exponent nonlinearity to the equations with variable-exponent nonlinearity.

1. Introduction

In this paper, we are concerned with the following wave equation with strong damping and variable-exponent nonlinearity

(1) $\quad u_{tt}(x,t) - \Delta u(x,t) - \Delta u_t(x,t) = |u(x,t)|^{p(x)-2} u(x,t) \qquad$ in $\Omega \times (0,T)$,

(2) $\quad u(x,t) = 0 \qquad$ on $\partial\Omega \times (0,T)$,

(3) $\quad u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x) \qquad$ on Ω,

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$, and the exponent $p(\cdot)$ is a given measurable function.

During the past decades, the following wave equations

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = |u|^{p-2} u$$

have been studied extensively on existence, nonexistence, stability, and blow up of solutions [5, 10, 12, 15, 17, 26–28]. When $\omega = \mu = 0$, Sattinger [27] discussed the existence of local as well as global solutions by introducing the concepts

---

of stable and unstable sets. Ball [5] established a finite time blow-up result of solutions with negative initial energy. When $\omega = 0$ and $\mu > 0$, Ikehata [15] gave a characterization of the existence of blow-up solutions for sufficiently small $\mu > 0$. Later, Esquivel-Avila [10] extended the result of [15] to the case of any $\mu > 0$. Gazzola and Squassina [12] proved the global existence and finite time blow up of solutions under the conditions $\omega \geq 0$ and $\mu > -\omega\lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ with homogeneous Dirichlet boundary conditions.

Recently, researchers have much interested in nonlinear models of hyperbolic, parabolic, and elliptic equations with variable-exponent nonlinearities [2, 22–25]. This kind of systems appears in electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media and image processing. We refer [1, 3, 18] for more details on these problems. Antontsev [2] studied a quasilinear equation of the form

$$u_{tt} - div(a(x,t)|\nabla u|^{r(x,t)-2}\nabla u) - \alpha\Delta u_t = b(x,t)|u|^{p(x,t)-2}u + f(x,t).$$

He proved the existence and blow up of weak solutions with negative initial energy under some conditions on a, b, f, r, and p. Messaoudi et al. [25] considered the following wave equations of variable-exponent nonlinearities

$$u_{tt}(x,t) - \Delta u(x,t) + a|u_t(x,t)|^{m(x)-2}u_t = b|u(x,t)|^{p(x)-2}u(x,t),$$

where a and b are positive constants, and $m(\cdot)$ and $p(\cdot)$ are given measurable functions. They proved the existence of a unique weak solution under suitable assumptions on the variable exponents $m(\cdot)$ and $p(\cdot)$ by using the Faedo-Galerkin method. Then, they established the finite time blow-up of solutions when the initial energy is negative. Regarding nonlinear wave equations with constant-exponent nonlinearities, we also refer [6, 13, 14, 20, 21] and references therein. Messaoudi and Talahmeh [23] discussed a quasilinear wave equation of the form

$$u_{tt}(x,t) - div(|\nabla u(x,t)|^{r(x)-2}\nabla u(x,t)) + a|u_t(x,t)|^{m(x)-2}u_t$$
$$= b|u(x,t)|^{p(x)-2}u(x,t).$$

They showed finite time blow-up results for the solution with negative initial energy and for certain solutions with positive initial energy. Motivated these results, we investigate finite time blow-up results of the solutions with positive initial energy as well as non-positive initial energy for problem (1)-(3). As far as we know, there are few works on wave equations with strong damping and variable-exponent nonlinearity. Moreover, this work extends the previous results for strongly damped wave equations with constant-exponent nonlinearity to the equations with variable-exponent nonlinearity.

The outline of this paper is as follows. In Section 2, we give materials needed for our work. In Section 3, we prove finite time blow-up results.

2. Preliminaries

In this section, we present some material needed for the statement and proof of our results. First, we give preliminary facts about Lebesque and Sobolev spaces with variable exponents (see [8, 9, 11]). Let $p : \Omega \to [1, \infty]$ be a measurable function. The Lebesque space $L^{p(\cdot)}(\Omega)$ with variable exponent $p(\cdot)$ is defined by

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable in } \Omega,$$

$$\int_\Omega |\lambda u(x)|^{p(x)} dx < \infty \text{ for some } \lambda > 0\}.$$

This space is a Banach space with the Luxembourg-type norm

$$||u||_{p(\cdot)} = \inf\{\lambda > 0 \mid \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\}.$$

In general, variable-exponent Lebesgue spaces are similar to classical Lebesgue spaces in many aspects (see [16]). As usual, $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ denote the inner product in the space $L^2(\Omega)$ and the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, respectively. $|| \cdot ||_q$ denotes the norm of the space $L^q(\Omega)$. For brevity, we denote $|| \cdot ||_2$ by $|| \cdot ||$.

Let us list some properties of the space $L^{p(\cdot)}(\Omega)$ which will be used in this work.

Lemma 2.1 ([7]). *If $p : \Omega \to [1, \infty)$ is a measurable function satisfying*

$$2 \leq p(x) < \infty \quad if \ n = 1, 2;$$

$$2 \leq ess \inf_{x \in \Omega} p(x) \leq p(x) \leq ess \sup_{x \in \Omega} p(x) < \frac{2n}{n-2} \ if \ n \geq 3,$$

then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.

Lemma 2.2 ([4]). *If*

$$1 < p_1 := ess \inf_{x \in \Omega} p(x) \leq p(x) \leq p_2 := ess \sup_{x \in \Omega} p(x) < \infty,$$

then

$$\min\{||u||_{p(\cdot)}^{p_1}, ||u||_{p(\cdot)}^{p_2}\} \leq \int_\Omega |u(x)|^{p(x)} dx \leq \max\{||u||_{p(\cdot)}^{p_1}, ||u||_{p(\cdot)}^{p_2}\}$$

for any $u \in L^{p(\cdot)}(\Omega)$.

Definition. Let $T > 0$. We say that a function u is a solution of problem (1)-(3) on $\Omega \times (0, T)$ if

$$u \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap C^2([0, T]; H^{-1}(\Omega))$$

with $u_t \in L^2(0, T; H_0^1(\Omega))$ and satisfies

$$(4) \qquad \langle u_{tt}(t), w \rangle + (\nabla u(t), \nabla w) + (\nabla u_t(t), \nabla w) = (|u(t)|^{p(\cdot)-2} u(t), w)$$

for any $w \in H_0^1(\Omega)$, and

$$u(0) = u_0 \text{ in } H_0^1(\Omega), \ u_t(0) = u_1 \text{ in } L^2(\Omega).$$

We state the well-posedness which can be established by the arguments of [19, 25].

Theorem 2.3. *Assume that $p(\cdot)$ satisfies*

$$2 < p_1 \leq p(x) \leq p_2 < \infty \ \text{ if } n = 1, 2;$$

$$2 < p_1 \leq p(x) \leq p_2 < \frac{2(n-1)}{n-2} \ \text{ if } n \geq 3,$$

where

$$p_1 := ess \ \inf_{x \in \Omega} p(x), \ \ p_2 := ess \ \sup_{x \in \Omega} p(x),$$

and the log-Hölder continuity condition:

(5) $|p(x) - p(y)| \leq -\dfrac{A}{\log|x-y|} \ \ \text{for a.e. } x, y \in \Omega, \ \ \text{with } |x - y| < \delta,$

here $A > 0$ and $0 < \delta < 1$. Then, for every $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, problem (1)-(3) has a unique local solution.

3. Finite time blow-up of solutions

In this section we prove that the solution to problem (1)-(3) blows up in finite time when the initial energy is positive as well as non-positive. In order to state and prove our results, we need the following lemma:

Lemma 3.1 ([17]). *Let $G(t)$ be a positive, twice differentiable function satisfying the inequality*

$$G(t)G''(t) - (1+\delta)(G'(t))^2 \geq 0 \ \text{ for } t > 0,$$

where δ is a positive constant. If $G(0) > 0$ and $G'(0) > 0$, then there exists a time $T_* \leq \frac{G(0)}{\delta G'(0)}$ such that

$$\lim_{t \to T_*^-} G(t) = +\infty.$$

We start by defining the energy of the solution to problem (1)-(3) as

(6) $E(t) = \dfrac{1}{2}||u_t(t)||^2 + \dfrac{1}{2}||\nabla u(t)||^2 - \displaystyle\int_\Omega \dfrac{|u(x,t)|^{p(x)}}{p(x)} dx.$

Then, multiplying (1) by u_t and integrating it over Ω, we find

(7) $E'(t) = -||\nabla u_t(t)||^2 \leq 0 \ \text{ for } 0 \leq t < T_{\max},$

where $T_{\max}$ is the maximal existence time of the solution u of problem (1)-(3). Moreover, we also get

(8) $E(t) + \displaystyle\int_0^t ||\nabla u_t(s)||^2 ds = E(0) \ \text{ for } 0 \leq t < T_{\max}.$

3.1. Blow up for non-positive initial energy

In this subsection, we show that the solution with non-positive initial energy blows up in finite time.

Theorem 3.2. *Let the conditions of Theorem 2.1 hold and $E(0) \leq 0$. Moreover, assume that $(u_0, u_1) > 0$ when $E(0) = 0$. Then the solution u of problem (1)-(3) blows up in finite time .*

Proof. By contradiction, suppose that the solution u is global. For any $T > 0$, we consider $G : [0, T] \rightarrow \mathbb{R}^+$ defined by

$$(9) \qquad G(t) = ||u(t)||^2 + \int_0^t ||\nabla u(s)||^2 ds + (T - t)||\nabla u_0||^2 + b(t + T_0)^2,$$

where $T_0 > 0$ and $b \geq 0$ to be specified later. Then

$$(10) \qquad\qquad\qquad\qquad G(t) > 0 \ \ \text{for } t \in [0, T]$$

and

$$G'(t) = 2(u(t), u_t(t)) + ||\nabla u(t)||^2 - ||\nabla u_0||^2 + 2b(t + T_0)$$

$$(11) \qquad\qquad = 2(u(t), u_t(t)) + 2\int_0^t (\nabla u(s), \nabla u_t(s))ds + 2b(t + T_0).$$

From (1), we obtain

$$G''(t) = 2||u_t(t)||^2 + 2\langle u_{tt}(t), u(t)\rangle + 2(\nabla u(t), \nabla u_t(t)) + 2b$$

$$(12) \qquad\qquad = 2||u_t(t)||^2 - 2||\nabla u(t)||^2 + 2\int_\Omega |u(x,t)|^{p(x)}dx + 2b.$$

By Cauchy-Schwartz inequality and (9), we get

$$\frac{(G'(t))^2}{4} = \left((u(t), u_t(t)) + \int_0^t (\nabla u(s), \nabla u_t(s))ds + b(t + T_0)\right)^2$$

$$\leq \left(||u(t)||^2 + \int_0^t ||\nabla u(s)||^2 ds + b(t + T_0)^2\right)$$

$$\times \left(||u_t(t)||^2 + \int_0^t ||\nabla u_t(s)||^2 ds + b\right)$$

$$= \left(G(t) - (T - t)||\nabla u_0||^2\right)\left(||u_t(t)||^2 + \int_0^t ||\nabla u_t(s)||^2 ds + b\right)$$

$$(13) \qquad \leq G(t)\left(||u_t(t)||^2 + \int_0^t ||\nabla u_t(s)||^2 ds + b\right).$$

From (12) and (13), we have the following differential inequality

$$(14) \qquad\qquad G(t)G''(t) - \frac{p_1 + 2}{4}(G'(t))^2 \geq G(t)F(t),$$

where

$$F(t) = -p_1||u_t(t)||^2 - 2||\nabla u(t)||^2 + 2\int_\Omega |u(x,t)|^{p(x)}dx$$

(15)
$$- (p_1 + 2)\int_0^t ||\nabla u_t(s)||^2 ds - p_1 b.$$

Using (6) and (8), we find

$$F(t) = -2p_1 E(t) + (p_1 - 2)||\nabla u(t)||^2 - 2p_1 \int_\Omega \frac{|u(x,t)|^{p(x)}}{p(x)}dx$$

$$+ 2\int_\Omega |u(x,t)|^{p(x)}dx - (p_1 + 2)\int_0^t ||\nabla u_t(s)||^2 ds - p_1 b$$

$$= -2p_1 E(0) + (p_1 - 2)||\nabla u(t)||^2 - 2p_1 \int_\Omega \frac{|u(x,t)|^{p(x)}}{p(x)}dx$$

$$+ 2\int_\Omega |u(x,t)|^{p(x)}dx + (p_1 - 2)\int_0^t ||\nabla u_t(s)||^2 ds - p_1 b$$

(16)
$$\geq -2p_1 E(0) + (p_1 - 2)||\nabla u(t)||^2 + (p_1 - 2)\int_0^t ||\nabla u_t(s)||^2 ds - p_1 b.$$

We now consider two cases $E(0) < 0$ and $E(0) = 0$.

Case 1: $E(0) < 0$.
Taking $0 < b \leq -2E(0)$, we have from (16) that

(17) $F(t) \geq 0.$

Case 2: $E(0) = 0$.
Choosing $b = 0$, we get from (16) that

(18) $F(t) \geq 0.$

Adapting (17) and (18) to (14), we infer

(19) $$G(t)G''(t) - \frac{p_1 + 2}{4}(G'(t))^2 \geq 0.$$

Now, it remains to show $G'(0) > 0$. When $E(0) < 0$, we take T_0 large enough such that

$$G'(0) = 2(u_0, u_1) + 2bT_0 > 0.$$

If $E(0) = 0$, the condition $(u_0, u_1) > 0$ gives

$$G'(0) = 2(u_0, u_1) > 0.$$

Thus, we conclude from Lemma 3.1 that

(20) $$\lim_{t \to T_*^-} G(t) = +\infty$$

for

(21) $$T_* \leq \frac{4G(0)}{(p_1 - 2)G'(0)} = \frac{2||u_0||^2 + 2T||\nabla u_0||^2 + 2bT_0^2}{(p_1 - 2)\Big((u_0, u_1) + bT_0\Big)}.$$

Thus, we deduce that

$$(22) \qquad T_* \leq \frac{2||u_0||^2 + 2bT_0^2}{(p_1 - 2)(u_0, u_1) + (p_1 - 2)bT_0 - 2||\nabla u_0||^2}.$$

From (9), (20) and (22), we have

$$(23) \qquad \lim_{t \to T_*^-} \left(||u(t)|| + \int_0^t ||\nabla u(s)||^2 ds \right) = +\infty.$$

This contradicts our assumption that the solution is global. Therefore, we conclude that the solution u to problem (1)-(3) blows up in finite time. $\qquad \square$

3.2. Blow up for positive initial energy

In this subsection, we establish a finite time blow-up result for certain solutions with positive energy. For this, we set

$$(24) \qquad \tilde{B} = \max\{1, B\}, \quad \xi_1 = \left(\frac{1}{\tilde{B}}\right)^{\frac{p_1}{p_1 - 2}}, \quad d = \frac{(p_1 - 2)\xi_1^2}{2p_1},$$

where B is the optimal constant of the embedding inequality

$$(25) \qquad ||v||_{p(\cdot)} \leq B||\nabla v|| \quad \text{for } v \in H_0^1(\Omega).$$

Next, we define a functional h by

$$(26) \qquad h(\xi) = \frac{1}{2}\xi^2 - \frac{\tilde{B}^{p_1}}{p_1}\xi^{p_1}.$$

We can easily check that h is continuous, $h(0) = 0$, and $\lim_{\xi \to +\infty} h(\xi) = -\infty$. Moreover, we see that h is increasing on $(0, \xi_1)$ and decreasing on (ξ_1, ∞). So, h has the maximum value $h(\xi_1) = d$.

Lemma 3.3. *Let u be the solution of problem (1)-(3). Assume that*

$$(27) \qquad E(0) < d \quad and \quad \xi_1 < ||\nabla u_0|| \leq \frac{1}{\tilde{B}}.$$

Then there exists a constant $\xi_* > \xi_1$ such that

$$(28) \qquad ||\nabla u(t)||^2 \geq \xi_*^2 \quad for \ 0 \leq t < T_{\max}.$$

Proof. From (6), Lemma 2.2, (25) and (24), we have

$$E(t) \geq \frac{1}{2}||\nabla u(t)||^2 - \frac{1}{p_1}\int_\Omega |u(x,t)|^{p(x)} dx$$

$$\geq \frac{1}{2}||\nabla u(t)||^2 - \frac{1}{p_1}\max\{||u(t)||_{p(\cdot)}^{p_1}, ||u(t)||_{p(\cdot)}^{p_2}\}$$

$$\geq \frac{1}{2}||\nabla u(t)||^2 - \frac{1}{p_1}\max\{\tilde{B}^{p_1}||\nabla u(t)||^{p_1}, \tilde{B}^{p_2}||\nabla u(t)||^{p_2}\}$$

$$(29) \qquad = g(||\nabla u(t)||),$$

where

$$g(\xi) = \frac{1}{2}\xi^2 - \frac{1}{p_1}\max\{\tilde{B}^{p_1}\xi^{p_1}, \tilde{B}^{p_2}\xi^{p_2}\}.$$

It is noted that

$$h(\xi) = g(\xi) \ \text{ for } 0 \leq \xi \leq \frac{1}{\tilde{B}},$$

where h is the function given in (26). Owing to $E(0) < d$, there exists $\xi_* > \xi_1$ such that

(30) $$E(0) = h(\xi_*).$$

Considering $t = 0$ in the inequality (29), we have from (27) and (30) that

(31) $$h(\xi_*) = E(0) \geq g(||\nabla u_0||) = h(||\nabla u_0||).$$

Since h is decreasing on (ξ_1, ∞), we see

(32) $$\xi_* \leq ||\nabla u_0||.$$

From (27), we also know

(33) $$\xi_* \leq \frac{1}{\tilde{B}}.$$

Now, we want to show that

(34) $$||\nabla u(t)|| \geq \xi_* \ \text{ for all } t \in [0, T_{\max}).$$

By contradiction, suppose that there exists $t_0 \in [0, T_{\max})$ such that

(35) $$||\nabla u(t_0)|| < \xi_*.$$

Because the solution u is continuous in t, there exists $t_1 > 0$ such that

(36) $$\xi_1 < ||\nabla u(t_1)|| < \xi_*.$$

Noting that h is decreasing on (ξ_1, ∞), we have from (33), (36) and (29) that

(37) $$E(0) = h(\xi_*) < h(||\nabla u(t_1)||) = g(||\nabla u(t_1)||) \leq E(t_1) \leq E(0),$$

we used the fact E is nonincreasing in the last inequality. But, this is contradiction. Thus we complete the proof. $\square$

Theorem 3.4. *Under the conditions of Lemma 3.2, the solution u of problem (1)-(3) blows up in finite time.*

Proof. Suppose that the solution u is global. For any $T > 0$, we consider the function G defined in (9). Then, (10), (14), (15), and (16) hold. First, we show that the function $F(t)$ given in (15) is non-negative. Indeed, the fact that the solution u is continuous on $[0, T]$ and Lemma 3.2 ensure the existence of $\epsilon > 0$ satisfying

(38) $$\xi_1^2 + \epsilon < \xi_*^2 \leq ||\nabla u(t)||^2 \ \text{ for all } t \in [0, T].$$

From (16), (38), and (24), we observe

(39) $$F(t) > -2p_1 d + (p_1 - 2)(\xi_1^2 + \epsilon) - p_1 b = (p_1 - 2)\epsilon - p_1 b.$$

Choosing $b > 0$ sufficiently small such that $(p_1 - 2)\epsilon - p_1 b \geq 0$, we obtain

(40) $$F(t) > 0.$$

From (40) and (14), we have

$$(41) \qquad G(t)G''(t) - \frac{p_1 + 2}{4}(G'(t))^2 > 0.$$

Choosing T_0 large enough such that

$$G'(0) = 2(u_0, u_1) + 2bT_0 > 0,$$

we can get

$$G'(0) > 0.$$

The remainder of the proof can be established by repeating the steps (20) to (23) of the proof of Theorem 3.1.   □

References

[1] R. Aboulaich, D. Meskine, and A. Souissi, *New diffusion models in image processing*, Comput. Math. Appl. **56** (2008), no. 4, 874–882. https://doi.org/10.1016/j.camwa.2008.01.017

[2] S. Antontsev, *Wave equation with $p(x,t)$-Laplacian and damping term: existence and blow-up*, Differ. Equ. Appl. **3** (2011), no. 4, 503–525. https://doi.org/10.7153/dea-03-32

[3] S. Antontsev and S. Shmarev, *Blow-up of solutions to parabolic equations with non-standard growth conditions*, J. Comput. Appl. Math. **234** (2010), no. 9, 2633–2645. https://doi.org/10.1016/j.cam.2010.01.026

[4] _____, *Evolution PDEs with nonstandard growth conditions*, Atlantis Studies in Differential Equations, 4, Atlantis Press, Paris, 2015. https://doi.org/10.2991/978-94-6239-112-3

[5] J. M. Ball, *Remarks on blow-up and nonexistence theorems for nonlinear evolution equations*, Quart. J. Math. Oxford Ser. (2) **28** (1977), no. 112, 473–486. https://doi.org/10.1093/qmath/28.4.473

[6] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and P. Martinez, *Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term*, J. Differential Equations **203** (2004), no. 1, 119–158. https://doi.org/10.1016/j.jde.2004.04.011

[7] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, 2017, Springer, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-18363-8

[8] D. E. Edmunds and J. Rákosník, *Sobolev embeddings with variable exponent*, Studia Math. **143** (2000), no. 3, 267–293. https://doi.org/10.4064/sm-143-3-267-293

[9] _____, *Sobolev embeddings with variable exponent. II*, Math. Nachr. **246/247** (2002), 53–67.

[10] J. A. Esquivel-Avila, *The dynamics of a nonlinear wave equation*, J. Math. Anal. Appl. **279** (2003), no. 1, 135–150. https://doi.org/10.1016/S0022-247X(02)00701-1

[11] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), no. 2, 424–446. https://doi.org/10.1006/jmaa.2000.7617

[12] F. Gazzola and M. Squassina, *Global solutions and finite time blow up for damped semilinear wave equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), no. 2, 185–207. https://doi.org/10.1016/j.anihpc.2005.02.007

[13] V. Georgiev and G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source terms*, J. Differential Equations **109** (1994), no. 2, 295–308. https://doi.org/10.1006/jdeq.1994.1051

[14] T. G. Ha, *Blow-up for semilinear wave equation with boundary damping and source terms*, J. Math. Anal. Appl. **390** (2012), no. 1, 328–334. `https://doi.org/10.1016/j.jmaa.2012.01.037`

[15] R. Ikehata, *Some remarks on the wave equations with nonlinear damping and source terms*, Nonlinear Anal. **27** (1996), no. 10, 1165–1175. `https://doi.org/10.1016/0362-546X(95)00119-G`

[16] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41(116)** (1991), no. 4, 592–618.

[17] H. A. Levine, *Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$*, Trans. Amer. Math. Soc. **192** (1974), 1–21. `https://doi.org/10.2307/1996814`

[18] S. Lian, W. Gao, C. Cao, and H. Yuan, *Study of the solutions to a model porous medium equation with variable exponent of nonlinearity*, J. Math. Anal. Appl. **342** (2008), no. 1, 27–38. `https://doi.org/10.1016/j.jmaa.2007.11.046`

[19] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Second ed. Dunod, Paris, 2002.

[20] S. A. Messaoudi, *Blow up in a nonlinearly damped wave equation*, Math. Nachr. **231** (2001), 105–111.

[21] _____, *Global nonexistence in a nonlinearly damped wave equation*, Appl. Anal. **80** (2001), no. 3-4, 269–277. `https://doi.org/10.1080/00036810108840993`

[22] _____, *On the decay of solutions of a damped quasilinear wave equation with variable-exponent nonlinearities*, Math. Meth. Appl. Sci. **43** (2020), 5114–5126.

[23] S. A. Messaoudi and A. A. Talahmeh, *Blowup in solutions of a quasilinear wave equation with variable-exponent nonlinearities*, Math. Methods Appl. Sci. **40** (2017), no. 18, 6976–6986. `https://doi.org/10.1002/mma.4505`

[24] _____, *A blow-up result for a nonlinear wave equation with variable-exponent nonlinearities*, Appl. Anal. **96** (2017), no. 9, 1509–1515. `https://doi.org/10.1080/00036811.2016.1276170`

[25] S. A. Messaoudi, A. A. Talahmeh, and J. H. Al-Smail, *Nonlinear damped wave equation: existence and blow-up*, Comput. Math. Appl. **74** (2017), no. 12, 3024–3041. `https://doi.org/10.1016/j.camwa.2017.07.048`

[26] L. E. Payne and D. H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math. **22** (1975), no. 3-4, 273–303. `https://doi.org/10.1007/BF02761595`

[27] D. H. Sattinger, *On global solution of nonlinear hyperbolic equations*, Arch. Rational Mech. Anal. **30** (1968), 148–172. `https://doi.org/10.1007/BF00250942`

[28] G. F. Webb, *Existence and asymptotic behavior for a strongly damped nonlinear wave equation*, Canadian J. Math. **32** (1980), no. 3, 631–643. `https://doi.org/10.4153/CJM-1980-049-5`

Sun-Hye Park
Office for Education Accreditation
Pusan National University
Busan 46241, Korea
Email address: `sh-park@pusan.ac.kr`