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MONOIDAL FUNCTORS AND EXACT SEQUENCES OF GROUPS FOR HOPF QUASIGROUPS

José N. Alonso Álvarez, José M. Fernández Vilaboa, and Ramón González Rodríguez

Abstract. In this paper we introduce the notion of strong Galois H-progenerator object for a finite cocommutative Hopf quasigroup H in a symmetric monoidal category C. We prove that the set of isomorphism classes of strong Galois H-progenerator objects is a subgroup of the group of strong Galois H-objects introduced in [3]. Moreover, we show that strong Galois H-progenerator objects are preserved by strong symmetric monoidal functors and, as a consequence, we obtain an exact sequence involving the associated Galois groups. Finally, to the previous functors, if H is finite, we find exact sequences of Picard groups related with invertible left H-(quasi)modules and an isomorphism $Pic({}_{\mathsf{H}}\mathsf{Mod}) \cong Pic(\mathsf{C}) \oplus G(H^{*})$ where $Pic({}_{\mathsf{H}}\mathsf{Mod})$ is the Picard group of the category of left H-modules, $Pic(\mathsf{C})$ the Picard group of C, and $G(H^{*})$ the group of group-like morphisms of the dual of H.

1. Introduction

Let R be a commutative ring with unit and let H be a Hopf algebra in the category ${}_{\mathsf{R}}\mathsf{Mod}$ of left R-modules. In [3] we extend the construction of the classical group of Galois H-objects to the non-associative setting of Galois objects associated to a Hopf quasigroup. More concretely, if C is a symmetric monoidal category with equalizers and H is a Hopf quasigroup in C, in the quoted paper we introduce the notions of Galois H-object and strong Galois H-object proving that, when H is cocommutative and faithfully flat, the set of isomorphism classes of strong Galois H-objects, denoted by $Gal^{s}(H)$, is a commutative group. If H is a Hopf algebra it is easy to show that strong Galois H-objects and Galois H-objects are the same thing and then in the

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associative setting $Gal^s(H)$ is the classical group of Galois H-objects (see [10] for $\mathsf{C} = {}_{\mathsf{R}}\mathsf{Mod}$).

On the other hand, it is well-known that in the Hopf algebra setting there exists exact sequences linking Galois groups, Picard groups, and groups of group-like elements. Some of these sequences are obtained using K-theoretical tools applied to monoidal functors (see [10] and [12]) as, for example, the extension of scalars functor associated to a homomorphism of commutative rings. The main motivation of this paper is to obtain similar exact sequences for Hopf quasigroups and monoidal functors between symmetric monoidal categories.

Let C be a symmetric monoidal category with equalizers and let H be a Hopf quasigroup in C. In [3] we introduce the notion of Galois H-object as a right H-comodule magma $\mathbb{A} = (A, \rho_A)$ such that A is faithfully flat and the canonical morphism $\gamma_A = (\mu_A \otimes id_H) \circ (id_A \otimes \rho_A) : A \otimes A \to A \otimes H$ is an isomorphism. Moreover, if $f_A = \gamma_A^{-1} \circ (\eta_A \otimes id_H) : H \to A^e$ is a morphism of unital magmas, the pair $\mathbb{A}$ is a strong Galois H-object. The first problem that we find for these objects when we work with monoidal functors is the following: assume that $F : \mathsf{C} \to \mathsf{D}$ is a strong symmetric monoidal functor, then $F(H)$ is a Hopf quasigroup in D but, if $\mathbb{A}$ is a strong Galois H-object in C, $\mathbb{F}(\mathbb{A})$ it is not a strong Galois $F(H)$-object in D because F does not preserve faithfully flat objects. To avoid this obstacle, in this paper we will assume that C admits coequalizers and, using the notion of progenerator in a symmetric monoidal category (for example, a left R-module P is a progenerator in ${}_{\mathsf{R}}\mathsf{Mod}$ if and only if P is finitely generated, projective and faithful), we introduce the notions of Galois H-progenerator object and strong Galois H-progenerator object. Every progenerator in C is a faithfully flat object and, as a consequence, if C admits coequalizers, every Galois H-progenerator object is a Galois H-object, and every strong Galois H-progenerator object is a strong Galois H-object. Moreover, if H is finite we have that H is a progenerator in C. As a consequence, the set of isomorphism classes of strong Galois H-progenerator objects forms a subgroup of $Gal^s(H)$ denoted by $Gal^{sp}(H)$. Moreover, if $F : \mathsf{C} \to \mathsf{D}$ is a strong symmetric monoidal functor, C and D admit coequalizers, and F preserves coequalizers, we obtain that F preserves progenerators. Therefore, under the previous conditions, we can prove that there exists a group morphism between $Gal^{sp}(H)$ and $Gal^{sp}(F(H))$, that cames from a product preserving functor between categories with product, and also an exact sequence

$$Aut(\mathbb{H}) \to Aut(\mathbb{F}(\mathbb{H})) \to K_1\Phi(G(F)) \to Gal^{sp}(H) \to Gal^{sp}(F(H)).$$

As was proved in [3], the group $Aut(\mathbb{H})$ of automorphisms of $\mathbb{H} = (H, \delta_H)$ as H-comodule magma can be identified with the group $G(H^*)$ of group-like morphisms of H^* when H is finite. Then, as a consequence, we can assure that there exists an exact sequence

$$G(H^*) \to G(F(H)^*) \to K_1\Phi(G(F)) \to Gal^{sp}(H) \to Gal^{sp}(F(H)).$$

Finally, in this paper we also study the exact sequences associated to invertible left modules and quasimodules for a Hopf quasigroup H. Firstly we obtain a generalization of the group isomorphism constructed by Caenepeel in [9] for Hopf algebras in ${}_{\mathsf{R}}\mathsf{Mod}$ (see also [12] for the monoidal version), i.e., we prove that, if H is a finite cocommutative Hopf quasigroup, there exists an isomorphism $Pic({}_{\mathsf{H}}\mathsf{Mod}) \cong Pic(\mathsf{C}) \oplus G(H^*)$ where $Pic({}_{\mathsf{H}}\mathsf{Mod})$ is the Picard group of the category of left H-modules and $Pic(\mathsf{C})$ the Picard group of C. Secondly we also prove that if C and D are symmetric monoidal categories, $F : \mathsf{C} \to \mathsf{D}$ is a strong symmetric monoidal functor, and H is a cocommutative Hopf quasigroup in C, there exists two exact sequences

$$Aut_{\mathsf{C}}(K) \to Aut_D(I) \to K_1\Phi(QPic(F)) \to Pic({}_{\mathsf{H}}\mathsf{QMod}) \to Pic({}_{\mathsf{F(H)}}\mathsf{QMod}),$$

$$Aut_{\mathsf{C}}(K) \to Aut_D(I) \to K_1\Phi(Pic(F)) \to Pic({}_{\mathsf{H}}\mathsf{Mod}) \to Pic({}_{\mathsf{F(H)}}\mathsf{Mod}),$$

where $Pic({}_{\mathsf{H}}\mathsf{QMod})$ is the Picard group of the category of left H-quasimodules and $Aut_{\mathsf{C}}(K)$, $Aut_D(I)$ are the groups of automorphisms of the unit objects of C and D respectively.

2. Progenerators and monoidal functors

Throughout this paper C is a symmetric monoidal category where $\otimes$ denotes the tensor product, K the unit object and c the isomorphism of symmetry. Without loss of generality, by the coherence theorems, we can assume the monoidal structure of C strict (see [13, Theorem XI.5.3]). Then, in this paper, we omit explicitly the associativity and unit constraints working as they were all identities.

We denote the class of objects of C by $|\mathsf{C}|$ and for each object M in the category C, the identity morphism by $id_M : M \to M$. For simplicity of notation, given objects M, N and P in C and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$. We will say that $P \in |\mathsf{C}|$ is flat if the endofunctor $P \otimes - : \mathsf{C} \to \mathsf{C}$, equivalently $- \otimes P : \mathsf{C} \to \mathsf{C}$, preserves equalizers. If moreover $P \otimes -$ reflects isomorphisms we say that P is faithfully flat. Similarly, we will say that P is coflat if $P \otimes - : \mathsf{C} \to \mathsf{C}$, equivalently $- \otimes P : \mathsf{C} \to \mathsf{C}$, preserves coequalizers.

By a unital magma in C we understand a triple $A = (A, \eta_A, \mu_A)$ where A is an object in C and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in C such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$. If μ_A is associative, i.e., $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$, the unital magma will be called a monoid in C. For any unital magma A, we will denote by $\overline{A}$ the opposite unital magma $(A, \eta_{\overline{A}} = \eta_A, \mu_{\overline{A}} = \mu_A \circ c_{A,A})$. Given two unital magmas (monoids) $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is a morphism of unital magmas (monoids) if $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$. By duality, a counital comagma in C is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in C

and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in C such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$. If δ_D is coassociative, i.e., $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$, the counital comagma will be called a comonoid. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are counital comagmas (comonoids), $f : D \to E$ is morphism of counital magmas (comonoids) if $(f \otimes f) \circ \delta_D = \delta_E \circ f$ and $\varepsilon_E \circ f = \varepsilon_D$.

If A, B are unital magmas (monoids) in C, the object $A \otimes B$ is a unital magma (monoid) in C where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. With A^e we will denote the unital magma $\overline{A} \otimes A$. In a dual way, if D, E are counital comagmas (comonoids) in C, $D \otimes E$ is a counital comagma (comonoid) in C where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

Let A be a monoid. The pair (M, ϕ_M) is a right A-module if M is an object in C and $\phi_M : M \otimes A \to M$ is a morphism in C satisfying $\phi_M \circ (M \otimes \eta_A) = id_M$, $\phi_M \circ (\phi_M \otimes B) = \phi_M \circ (M \otimes \mu_A)$. Given two right A-modules (M, ϕ_M) and (N, ϕ_N), $f : M \to N$ is a morphism of right A-modules if $\phi_N \circ (f \otimes A) = f \circ \phi_M$. In the following, we will denote the category of right A-modules by $\mathsf{Mod_A}$. In a similar way we can define the notions of left A-module (we denote the left action by φ_M) and morphism of left A-modules. In this case the category of left A-modules will be denoted by $_{\mathsf{A}}\mathsf{Mod}$. Finally, note that K is a monoid and in this case we can identify the categories $\mathsf{Mod_K}$ and $_{\mathsf{K}}\mathsf{Mod}$ with C.

Let A, B be monoids. An A-B-bimodule is a triple (M, φ_M, ϕ_M) where (M, φ_M) is a left A-module, (M, ϕ_M) is a right B-module and $\varphi_M \circ (A \otimes \phi_M) = \phi_M \circ (\varphi_M \otimes B)$. With the obvious morphisms A-B-bimodules constitute a category that we will denote by $_{\mathsf{A}}\mathsf{Bimod_B}$.

Assume that C admits coequalizers and let A be a monoid in C, let (M, ϕ_M) be a right A-module and let (N, φ_N) be a left A-module. We define the object $M \otimes_A N$ by the following coequalizer diagram in C:

$$\begin{tikzcd} M \otimes A \otimes N \arrow[r, shift left, "\phi_M \otimes N"] \arrow[r, shift right, "M \otimes \varphi_N"'] & M \otimes N \arrow[r, "n_M^N"] & M \otimes_A N. \end{tikzcd} \tag{1}$$

Then, if $N = A$ and $\varphi_A = \mu_A$ we obtain an isomorphism $r_M : M \otimes_A A \to M$. This isomorphism is the unique morphism such that $\phi_M = r_M \circ n_M^A$. Analogously, for a left A-module (N, φ_N), the object $A \otimes_A N$ is isomorphic to N, i.e., there exists a unique isomorphism $l_N : A \otimes_A N \to N$ such that $\varphi_N = l_M \circ n_A^N$. Moreover, if (N, φ_N, ϕ_N) is an object in $_{\mathsf{A}}\mathsf{Bimod_B}$ and B is coflat, we have that $M \otimes_A N$ is a right B-module where the action $\phi_{M\otimes_A N} : M \otimes_A N \otimes B \to M \otimes_A N$ is defined as the unique morphism such that $\phi_{M\otimes_A N} \circ (n_M^N \otimes B) = n_M^N \circ (M \otimes \phi_N)$. Similarly, if (P, φ_P) is a left B-module and A is coflat, the object $N \otimes_B P$ is a left A-module where the left action is defined as the unique morphism $\varphi_{N\otimes_B P} : A \otimes N \otimes_B P \to N \otimes_B P$ such that $\varphi_{N\otimes_B P} \circ (A \otimes n_N^P) = n_N^P \circ (\varphi_N \otimes P)$. Finally, if M and N are coflat, there exists a unique isomorphism $a_{M,N,P} : M \otimes_A (N \otimes_B P) \to (M \otimes_A N) \otimes_B P$ such that

$$a_{M,N,P} \circ n_M^{N\otimes_B P} \circ (M \otimes n_N^P) = n_{M\otimes_A N}^P \circ (n_M^N \otimes P).$$

Definition 2.1. An object P in C is said to be finite if there exists $P^* \in |\mathsf{C}|$ such that $P \otimes - \dashv P^* \otimes -$. In what follows we will denote the unit of the previous adjunction by α_P and the counit by β_P. Usually the object P^* is called the dual of P. Note that, by the triangular identities for the adjunction $P \otimes - \dashv P^* \otimes -$, the identities

$$(2)\quad (\beta_P(K)\otimes P)\circ(P\otimes\alpha_P(K)) = id_P, \quad (P^*\otimes\beta_P(K))\circ(\alpha_P(K)\otimes P^*) = id_{P^*},$$

hold.

If P and Q are finite, $P \otimes Q$ is finite and $(P \otimes Q)^* = Q^* \otimes P^*$. For the associated adjunction $P \otimes Q \otimes - \dashv Q^* \otimes P^* \otimes -$ the unit is defined by $\alpha_{P\otimes Q} = (Q^* \otimes \alpha_P(K) \otimes Q \otimes -) \circ (\alpha_Q(K) \otimes -)$ and the counit by $\beta_{P\otimes Q} = (\beta_P(K) \otimes -) \circ (P \otimes \beta_Q(K) \otimes P^* \otimes -)$ (see [2, Proposition 3.4]).

On the other hand, if P is a finite object, P^* is finite and $P^{**} = P$ because $P^* \otimes - \dashv P \otimes -$ with $\alpha_{P^*} = (c_{P^*,P} \otimes -) \circ \alpha_P$ and $\beta_{P^*} = \beta_P \circ (c_{P^*,P} \otimes -)$ [2, Proposition 3.5].

As a consequence, if P is a finite object, $P \otimes - \dashv P^* \otimes - \dashv P \otimes -$ and then P is flat and coflat.

Let P and Q be finite objects in C and let $f : P \to Q$ be a morphism between them. There exists a new morphism $f^* : Q^* \to P^*$, called the dual of f, where

$$(3)\qquad f^* = (P^* \otimes \beta_Q(K)) \circ (P^* \otimes f \otimes Q^*) \circ (\alpha_P(K) \otimes Q^*).$$

Obviously, if P and Q are finite $f^{**} = f$.

If P is a finite object and we denote by $E(P)$ the tensor product $P^* \otimes P$, using the properties of the adjunction it is easy to show that $E(P)$ is a monoid with unit $\eta_{E(P)} = \alpha_P(K)$ and $\mu_{E(P)} = P^* \otimes \beta_P(K) \otimes P$. Moreover, P is a right $E(P)$-module with action $\phi_P = \beta_P(K) \otimes P$ and P^* is a left $E(P)$-module with action $\varphi_{P^*} = P^* \otimes \beta_P(K)$. Then, if C admits coequalizers, we can define the object $P \otimes_{E(P)} P^*$ as in (1) by the coequalizer diagram

$$\begin{tikzcd} P \otimes E(P) \otimes P^* \arrow[r, shift left, "\phi_P \otimes P^*"] \arrow[r, shift right, "P \otimes \varphi_{P^*}"'] & P \otimes P^* \arrow[r, "n_P^{P^*}"] & P \otimes_{E(P)} P^*. \end{tikzcd}$$

The morphism $\beta_P(K)$ satisfies that $\beta_P(K) \circ (\phi_P \otimes P^*) = \beta_P(K) \circ (P \otimes \varphi_{P^*})$ and then there exists a unique morphism $\nabla_P : P \otimes_{E(P)} P^* \to K$ such that

$$(4)\qquad \nabla_P \circ n_P^{P^*} = \beta_P(K).$$

Definition 2.2. Let P be a finite object in a symmetric monoidal category C with coequalizers. We will say that P is a progenerator if ∇_P is an isomorphism.

Thus P is a progenerator if and only if

$$P \otimes E(P) \otimes P^* \overset{\phi_P \otimes P^*}{\underset{P \otimes \varphi_{P^*}}{\rightrightarrows}} P \otimes P^* \xrightarrow{\beta_P(K)} K$$

is a coequalizer diagram. Also, by [4, 3.3.14] we know that ∇_P is an isomorphism if and only if there exists a morphism $\Delta_P : K \to P \otimes_{E(P)} P^*$ such that

$$\nabla_P \circ \Delta_P = id_K. \tag{5}$$

The trivial example of progenerator in C is the unit object K. If P and Q are progenerators so is $P \otimes Q$ [2, Proposition 3.4]. Moreover, by [2, Proposition 3.5] we have that P^* is a progenerator if P is a progenerator. On the other hand, for any progenerator P the functors $P \otimes_{E(P)} - : {}_{\mathsf{E(P)}}\mathsf{Mod} \to \mathsf{C}$ and $P^* \otimes - : \mathsf{C} \to {}_{\mathsf{E(P)}}\mathsf{Mod}$ induce a categorical equivalence. Therefore for any object M in C, $P \otimes_{E(P)} (P^* \otimes M) \cong M$ and, if N is a left $E(P)$-module, $(N \otimes_{E(P)} P^*) \otimes P \cong N \otimes_{E(P)} E(P) \cong N$. These properties imply that any progenerator is faithfully flat. Finally, if P and Q are progenerators and M is an object of C such that $P \otimes Q \cong M \otimes Q$ ($Q \otimes P \cong Q \otimes M$), we get that $P \cong M$ and then M is a progenerator.

Example 2.3. Let R be a commutative ring with unit. The category ${}_{\mathsf{R}}\mathsf{Mod}$, with tensor product $\otimes = \otimes_R$ and unit $K = R$, is an example of symmetric monoidal category with equalizers and coequalizers. By the results proved in [11], a left R-module P is a progenerator in the category ${}_{\mathsf{R}}\mathsf{Mod}$ if and only if P is finitely generated, projective and faithful.

Proposition 2.4. *Let P and Q be finite objects in a symmetric monoidal category C with coequalizers. Assume that there exist morphisms $f : P \to Q$ and $g : Q \to P$ such that $g \circ f = id_P$. Then, if P is a progenerator so is Q.*

Proof. Let $g^* : P^* \to Q^*$ be the dual morphism of g defined in (3). Then, the equality

$$n_Q^{Q^*} \circ (\phi_Q \otimes Q^*) \circ (f \otimes g^* \otimes f \otimes g^*) = n_Q^{Q^*} \circ (Q \otimes \varphi_Q) \circ (f \otimes g^* \otimes f \otimes g^*)$$

holds by the definition of $n_Q^{Q^*}$. Moreover, if we define the morphism $h : P \otimes P^* \to Q \otimes_{E(Q)} Q^*$ as $h = n_Q^{Q^*} \circ (f \otimes g^*)$, by (2) and the equality $g \circ f = id_P$, we have that

$$n_Q^{Q^*} \circ (\phi_Q \otimes Q^*) \circ (f \otimes g^* \otimes f \otimes g^*) = h \circ (\phi_P \otimes P^*)$$

and

$$n_Q^{Q^*} \circ (Q \otimes \varphi_Q) \circ (f \otimes g^* \otimes f \otimes g^*) = h \circ (P \otimes \varphi_{P^*}).$$

Therefore, $h\circ(\phi_P\otimes P^*) = h\circ(P\otimes\varphi_{P^*})$ and, as a consequence, there exists a unique

$$\omega : P\otimes_{E(P)} P^* \to Q\otimes_{E(Q)} Q^*$$

such that

$$\omega\circ n_P^{P^*} = h. \tag{6}$$

By (6), (4) (for Q and P), (2) and the equality $g\circ f = id_P$, we have the following identity:

$$\nabla_Q\circ\omega\circ n_P^{P^*} = \nabla_Q\circ n_Q^{Q^*}\circ(f\otimes g^*) = \beta_Q(K)\circ(f\otimes g^*) = \beta_P(K) = \nabla_P\circ n_P^{P^*}.$$

Thus

$$\nabla_Q\circ\omega = \nabla_P. \tag{7}$$

On the other hand, P is a progenerator and then there exists a morphism $\Delta_P : K\to P\otimes_{E(P)} P^*$ satisfying (5). Define $\Delta_Q : K\to Q\otimes_{E(Q)} Q^*$ as $\Delta_Q = \omega\circ\Delta_P$. Thus,

$$\nabla_Q\circ\Delta_Q = \nabla_Q\circ\omega\circ\Delta_P \overset{(7)}{=} \nabla_P\circ\Delta_P \overset{(5)}{=} id_K.$$

Therefore, $\nabla_Q\circ\Delta_Q = id_K$ and Q is a progenerator. □

Definition 2.5. Let D be a symmetric monoidal category with tensor product $\boxtimes$, unit object I and isomorphism of symmetry t. A functor $F : \mathsf{C}\to\mathsf{D}$ is said to be symmetric monoidal if there exist morphisms $\Phi_0 : I\to F(K)$ and $\Phi_{M,N} : F(M)\boxtimes F(N)\to F(M\otimes N)$ (natural in M and N) such that:

(i) $\Phi_{M\otimes N,L}\circ(\Phi_{M,N}\boxtimes F(L)) = \Phi_{M,N\otimes L}\circ(F(M)\boxtimes\Phi_{N,L})$ for M, N and L in C.

(ii) $\Phi_{M,K}\circ(F(M)\boxtimes\Phi_0) = id_{F(M)} = \Phi_{K,M}\circ(\Phi_0\boxtimes F(M))$ for M in C.

(iii) $F(c_{M,N})\circ\Phi_{M,N} = \Phi_{N,M}\circ t_{F(M),F(N)}$ for M and N in C.

The symmetric monoidal functor F is said to be strong if Φ_0 and $\Phi_{N,M}$ are isomorphisms for all M, N objects in C. Note that in this case, by (i) of the previous definition, we have the following identities:

$$(\Phi^{-1}_{M,N}\boxtimes F(L))\circ\Phi^{-1}_{M\otimes N,L} = (F(M)\boxtimes\Phi^{-1}_{N,L})\circ\Phi^{-1}_{M,N\otimes L}, \tag{8}$$

$$(F(M)\boxtimes\Phi_{N,L})\circ(\Phi^{-1}_{M,N}\boxtimes F(L)) = \Phi^{-1}_{M,N\otimes L}\circ\Phi_{M\otimes N,L}, \tag{9}$$

$$(\Phi_{M,N}\boxtimes F(L))\circ(F(M)\boxtimes\Phi^{-1}_{N,L}) = \Phi^{-1}_{M\otimes N,L}\circ\Phi_{M,N\otimes L}. \tag{10}$$

On the other hand, by (ii) of Definition 2.5 we obtain that

$$\Phi_{M,K} = F(M)\boxtimes\Phi_0^{-1},\quad \Phi_{K,M} = \Phi_0^{-1}\boxtimes F(M), \tag{11}$$

$$\Phi^{-1}_{M,K} = F(M)\boxtimes\Phi_0,\quad \Phi^{-1}_{K,M} = \Phi_0\boxtimes F(M), \tag{12}$$

and then

$$\Phi_{K,K} = \Phi_0^{-1}\boxtimes F(K) = F(K)\boxtimes\Phi_0^{-1}, \tag{13}$$

$$\Phi_{K,K}^{-1} = F(K) \boxtimes \Phi_0, \quad \Phi_{K,K}^{-1} = \Phi_0 \boxtimes F(K). \tag{14}$$

Finally, by (iii),

$$\begin{aligned} F(c_{M,N}) &= \Phi_{N,M} \circ t_{F(M),F(N)} \circ \Phi_{M,N}^{-1}, \\ t_{F(M),F(N)} &= \Phi_{N,M}^{-1} \circ F(c_{M,N}) \circ \Phi_{M,N}, \end{aligned} \tag{15}$$

hold.

Let $F : \mathsf{C} \to \mathsf{D}$ be a symmetric monoidal functor. If $A = (A, \eta_A, \mu_A)$ is a (monoid) unital magma in C, it is easy to prove that

$$F(A) = (F(A), \eta_{F(A)} = F(\eta_A) \circ \Phi_0, \mu_{F(A)} = F(\mu_A) \circ \Phi_{A,A}) \tag{16}$$

is a (monoid) unital magma in D. For (comonoids) counital comagmas we have a similar property, i.e., if $F : \mathsf{C} \to \mathsf{D}$ is a strong symmetric monoidal functor and $C = (C, \varepsilon_C, \delta_C)$ is a (comonoid) counital comagma in C, the triple

$$F(C) = (F(C), \varepsilon_{F(C)} = \Phi_0^{-1} \circ F(\varepsilon_C), \delta_{F(C)} = \Phi_{C,C}^{-1} \circ F(\delta_C)) \tag{17}$$

is a (comonoid) counital comagma in D.

On the other hand, if $f : A \to B$ is a morphism of unital magmas in C, so is $F(f) : F(A) \to F(B)$ in D. Similarly, if $g : C \to D$ is a morphism of counital comagmas in C, so is $F(g) : F(C) \to F(D)$ in D.

The following results were proved in [2] and they will be useful in the following sections.

Proposition 2.6 ([2, Propositions 6.2, 6.4])**.** *Let $F : \mathsf{C} \to \mathsf{D}$ be a strong symmetric monoidal functor. Then*:

(i) *If P is a finite object in C, $F(P)$ is a finite object in D. Moreover, $F(P^*) \cong F(P)^*$.*
(ii) *If P is a finite object in C, the monoids $F(E(P))$ and $E(F(P))$ are isomorphic in D.*
(iii) *If C and D admit coequalizers and F preserves coequalizers, F preserves progenerators.*

Proposition 2.7 ([2, Proposition 6.4])**.** *Let $F : \mathsf{C} \to \mathsf{D}$ be a symmetric monoidal functor. If $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$ are (monoids) unital magmas in C, then $\Phi_{A,B}$ is a morphism of (monoids) unital magmas in D.*

Similarly, by duality, we have the corresponding result for (comonoids) counital comagmas.

Proposition 2.8. *Let $F : \mathsf{C} \to \mathsf{D}$ be a strong symmetric monoidal functor. If $C = (C, \varepsilon_C, \delta_C)$ and $D = (D, \varepsilon_D, \delta_D)$ are (comonoids) counital comagmas in C, then $\Phi_{C,D}$ is a morphism of (comonoids) counital comagmas in D.*

3. Hopf quasigroups and Hopf coquasigroups

The notion of Hopf quasigroup is a generalization of the one of Hopf algebra and was introduced by Klim and Majid in [15] in order to understand the structure and relevant properties of the algebraic 7-sphere. They are not associative but the lack of this property is compensated by some axioms involving the antipode. Hopf quasigroups are particular instances of unital coassociative H-bialgebras (see [17]) and include the example of the enveloping algebra of a Malcev algebra (see [15]) as well as the notion of quasigroup algebra of an I.P. loop. Then, quasigroups unify I.P. loops and Malcev algebras in the same way that Hopf algebras unified groups and Lie algebras. The definition of these kind of objects in a monoidal setting is the following.

Definition 3.1. A Hopf quasigroup H in C is a unital magma (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(i) The morphisms ε_H and δ_H are morphisms of unital magmas (equivalently, η_H and μ_H are morphisms of comonoids), i.e., the following identities hold:
  (i-1) $\varepsilon_H \circ \eta_H = id_K$,
  (i-2) $\varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H$,
  (i-3) $\delta_H \circ \eta_H = \eta_H \otimes \eta_H$,
  (i-4) $\delta_H \circ \mu_H = \mu_{H\otimes H} \circ (\delta_H \otimes \delta_H)$.

(ii) There exists $\lambda_H : H \to H$ in C (called the antipode of H) such that:
  (ii-1) $$\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H)$$
  $$= \varepsilon_H \otimes H = \mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H),$$
  (ii-2) $$\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H)$$
  $$= H \otimes \varepsilon_H = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H).$$

If H is a Hopf quasigroup, the antipode is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariable:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H$$

([15, Proposition 4.2], and [16, Proposition 1]). Note that by (ii),

$$\mu_H \circ (\lambda_H \otimes id_H) \circ \delta_H = \mu_H \circ (id_H \otimes \lambda_H) \circ \delta_H = \varepsilon_H \otimes \eta_H. \tag{18}$$

A Hopf quasigroup H is cocommutative if $c_{H,H} \circ \delta_H = \delta_H$. In this case, as in the Hopf algebra setting, we have that $\lambda_H \circ \lambda_H = id_H$ (see [15, Proposition 4.3]).

Let H and B be Hopf quasigroups. We say that $f : H \to B$ is a morphism of Hopf quasigroups if it is a morphism of unital magmas and comonoids. In this case $\lambda_B \circ f = f \circ \lambda_H$ [1, Proposition 1.5].

Example 3.2. Let R be a commutative ring with unit and with $\frac{1}{2}$ and $\frac{1}{3}$ in R. A Malcev algebra $(M, [\ ,\ ])$ over R is a free module over R with a bilinear anticommutative operation $[\ ,\ ]$ on M satisfying that

$$[J(a,b,c),a] = J(a,b,[a,c]),$$

where $J(a,b,c) = [[a,b],c] - [[a,c],b] - [a,[b,c]]$ is the Jacobian in a,b,c. By the construction given in [18] a Hopf quasigroup structure arises from M in the following way: Consider the not necessarily associative algebra $U(M)$ defined as the quotient of $R\{M\}$, the free non-associative algebra on a basis of M, by the ideal $I(M)$ generated by the set

$$\{ab - ba - [a,b], (a,x,y) + (x,a,y), (x,a,y) + (x,y,a) : a,b \in M, x,y \in R\{M\}\},$$

where $(x,y,z) = (xy)z - x(yz)$ is the usual additive associator. Let $\{a_i : i \in \Lambda_M\}$ be a basis of M, $\leq$ an order in Λ_M and $\Omega_M = \{(i_1,\ldots,i_n) : i_i,\ldots i_n \in \Lambda_M, n \in \mathbb{N} \text{ and } i_1 \leq \ldots \leq i_n\}$. If $I = (i_1,\ldots,i_n) \in \Omega_M$, then we write $\overline{a}_I^M$ instead of $\overline{a}_{i_1}^M(\overline{a}_{i_2}^M(\cdots(\overline{a}_{i_{n-1}}^M\overline{a}_{i_n}^M)\cdots))$. As usual, if $n = 0$, then $I = \emptyset$ and $1_{U(M)} = \overline{a}_I^M$. With this notation, the set $\{\overline{a}_I^M : I \in \Omega_M\}$ is a basis of $U(M)$ (Poincaré-Birkhoff-Witt theorem for Malcev algebras [18, Theorem 2.1]). By [18, Proposition 4.1] and [15, Proposition 4.8], $U(M)$ is a cocommutative Hopf quasigroup structure with coproduct $\delta_{U(M)} : U(M) \to U(M) \otimes U(M)$ defined by $\delta_{U(M)}(x) = 1\otimes x + x\otimes 1$ for all $x \in M$, counit $\varepsilon_{U(M)} : U(M) \to K$ defined by $\varepsilon_{U(M)}(x) = 0$ for all $x \in M$ (both extended to $U(M)$ as algebra morphisms), and antipode $\lambda_{U(M)} : U(M) \to U(M)$, defined by $\lambda_{U(M)}(x) = -x$ for all $x \in M$ and extended to $U(M)$ as an antialgebra morphism.

Example 3.3. A quasigroup is a set Q together with a product such that for any two elements $u, v \in Q$ the equations $ux = v$, $xu = v$ and $uv = x$ have unique solutions in Q. A quasigroup L which contains an element e_L such that $ue_L = u = e_L u$ for every $u \in L$ is called a loop. A loop L is said to be a loop with the inverse property (for brevity an I.P. loop) if and only if, to every element $u \in L$, there corresponds an element $u^{-1} \in L$ such that the equations $u^{-1}(uv) = v = (vu)u^{-1}$ hold for every $v \in L$.

If L is an I.P. loop, it is easy to show (see [6]) that for all $u \in L$ the element u^{-1} is unique and $u^{-1}u = e_L = uu^{-1}$. Moreover, the mapping $u \to u^{-1}$ is an anti-automorphism of the I.P. loop L, i.e., $(uv)^{-1} = v^{-1}u^{-1}$.

Let R be a commutative ring with unit and L an IP loop. Then, by [15, Proposition 4.7], we know that the loop algebra

$$R[L] = \bigoplus_{u \in L} Ru$$

is a cocommutative Hopf quasigroup with product defined by the linear extension of the one defined in L and $\delta_{R[L]}(u) = u\otimes u$, $\varepsilon_{R[L]}(u) = 1_R$, $\lambda_{R[L]}(u) = u^{-1}$ on the basis elements.

Definition 3.4. A Hopf coquasigroup D in $\mathcal{C}$ is a monoid (D, η_D, μ_D) and a counital comagma $(D, \varepsilon_D, \delta_D)$ such that the following axioms hold:

(i) The morphisms η_D and μ_D are morphisms of counital comonoids.

(ii) There exists $\lambda_D : D \to D$ in C (called the antipode of D) such that:

(ii-1) $(\mu_D \otimes D) \circ (\lambda_D \otimes \delta_D) \circ \delta_D$
$$= \eta_D \otimes D = (\mu_D \otimes D) \circ (D \otimes ((\lambda_D \otimes D) \circ \delta_D)) \circ \delta_D,$$

(ii-2) $(D \otimes \mu_D) \circ (\delta_D \otimes \lambda_D) \circ \delta_D$
$$= D \otimes \eta_D = (D \otimes \mu_D) \circ (((D \otimes \lambda_D) \circ \delta_D) \otimes D) \circ \delta_D.$$

As in the case of Hopf quasigroups, the antipode is unique, antimultiplicative, anticomultiplicative, leaves the unit and the counit invariable and satisfies (18).

Proposition 3.5. *Assume that* C *admits coequalizers. Then every finite Hopf* (*co*)*quasigroup is a progenerator. As a consequence, every finite Hopf* (*co*)*quasigroup is faithfully flat.*

Proof. If H is a Hopf quasigroup we have that (i-1) of Definition 3.1 holds. Then, if H is finite, by Proposition 2.4, we obtain that H is a progenerator because so is K. For Hopf coquasigroups the proof is similar. □

It is easy to show that, if H is a finite (commutative) cocommutative Hopf (coquasigroup) quasigroup, its dual H^* is a (cocommutative) commutative finite Hopf (quasigroup) coquasigroup where:

$$\eta_{H^*} = (H^* \otimes \varepsilon_H) \circ \alpha_H(K),$$
$$\mu_{H^*} = (H^* \otimes \beta_H(K)) \circ (H^* \otimes H \otimes \beta_H(K) \otimes H^*) \circ (H^* \otimes \delta_H \otimes H^* \otimes H^*) \circ (\alpha_H(K) \otimes H^* \otimes H^*),$$
$$\varepsilon_{H^*} = \beta_H(K) \circ (\eta_H \otimes H^*),$$
$$\delta_{H^*} = (H^* \otimes H^* \otimes (\beta_H(K) \circ (\mu_H \otimes H^*))) \circ (H^* \otimes \alpha_H(K) \otimes H \otimes H^*) \circ (\alpha_H(K) \otimes H^*)$$

and the antipode is $\lambda_{H^*} = (\lambda_H)^*$.

Proposition 3.6. *Let* $F : \mathsf{C} \to \mathsf{D}$ *be a strong symmetric monoidal functor. If*
$$H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H)$$
is a Hopf quasigroup (*Hopf coquasigroup*) *in* C*, then* $F(H)$ *with unit and product defined as in* (16)*, counit and coproduct defined as in* (17)*, and antipode* $\lambda_{F(H)} = F(\lambda_H)$*, is a Hopf quasigroup* (*Hopf coquasigroup*) *in* D*. If H is cocommutative* (*commutative*)*, so is* $F(H)$*.*

Proof. We will prove the proposition for Hopf quasigroups. The proof for Hopf coquasigroups is dual and we leave the details to the reader. First note that, $(F(H), \eta_{F(H)} = F(\eta_H) \circ \Phi_0, \mu_{F(H)} = F(\mu_H) \circ \Phi_{H,H})$ is a unital magma in D and $(F(H), \varepsilon_{F(H)} = \Phi_0^{-1} \circ F(\varepsilon_H), \delta_{F(H)} = \Phi_{H,H}^{-1} \circ F(\delta_H))$ is a comonoid in D. Also, by Proposition 2.7, we have that $\Phi_{H,H}$ is a morphism of unital magmas, i.e., $\Phi_{H,H} \circ \eta_{F(H) \boxtimes F(H)} = \eta_{F(H \otimes H)}$, and

$$\mu_{F(H \otimes H)} \circ (\Phi_{H,H} \boxtimes \Phi_{H,H}) = \Phi_{H,H} \circ \mu_{F(H) \boxtimes F(H)} \tag{19}$$

hold.

Trivially, $\varepsilon_{F(H)} \circ \eta_{F(H)} = id_I$ and then (i-1) of Definition 3.1 holds. By (i-2) of Definition 3.1 for H, the naturality of Φ, and (13) we have that

$$\begin{aligned}\varepsilon_{F(H)} \circ \mu_{F(H)} &= \Phi_0^{-1} \circ F(\varepsilon_H \circ \mu_H) \circ \Phi_{H,H} = \Phi_0^{-1} \circ F(\varepsilon_H \otimes \varepsilon_H) \circ \Phi_{H,H} \\ &= \Phi_0^{-1} \circ \Phi_{K,K} \circ (F(\varepsilon_H) \boxtimes F(\varepsilon_H)) = \varepsilon_{F(H)} \boxtimes \varepsilon_{F(H)}\end{aligned}$$

and then (i-2) of Definition 3.1 holds for $F(H)$. In a similar way we obtain that (i-3) of Definition 3.1 holds for $F(H)$ using (i-3) of Definition 3.1 for H, the naturality of Φ, and (14). On the other hand,

$$\begin{aligned}&\mu_{F(H)\boxtimes F(H)} \circ (\delta_{F(H)} \boxtimes \delta_{F(H)}) \\ &= \Phi_{H,H}^{-1} \circ F(\mu_{H\otimes H}) \circ \Phi_{H\otimes H,H\otimes H} \circ (\Phi_{H,H} \otimes \Phi_{H,H}) \circ (\delta_{F(H)} \boxtimes \delta_{F(H)}) \;\; ((19)) \\ &= \Phi_{H,H}^{-1} \circ F(\mu_{H\otimes H} \circ (\delta_H \otimes \delta_H)) \circ \Phi_{H,H} \;\; (\text{naturality of } \Phi) \\ &= \Phi_{H,H}^{-1} \circ F(\delta_H \circ \mu_H) \circ \Phi_{H,H} \;\; ((\text{i-4}) \text{ of Definition 3.1 for } H) \\ &= \delta_{F(H)} \circ \mu_{F(H)} \;\; (\text{definitions of } \delta_{F(H)} \text{ and } \mu_{F(H)})\end{aligned}$$

and $\varepsilon_{F(H)}$ and $\delta_{F(H)}$ are morphisms of unital magmas.

To obtain the proof of (ii) of Definition 3.1 for $F(H)$, we only prove the first identity of (ii-1) for $F(H)$. The proofs for the other identities are similar and we leave the details to the reader. Indeed:

$$\begin{aligned}&\mu_{F(H)} \circ (\lambda_{F(H)} \boxtimes \mu_{F(H)}) \circ (\delta_{F(H)} \boxtimes F(H)) \\ &= F(\mu_H \circ (H \otimes \mu_H)) \circ \Phi_{H,H\otimes H} \circ (F(H) \boxtimes \Phi_{H,H}) \\ &\quad \circ ((\Phi_{H,H}^{-1} \circ F((\lambda_H \otimes H) \circ \delta_H)) \boxtimes F(H)) \;\; (\text{naturality of } \Phi) \\ &= F(\mu_H \circ (H \otimes \mu_H)) \circ \Phi_{H\otimes H,H} \circ (F((\lambda_H \otimes H) \circ \delta_H) \boxtimes F(H)) \;\; ((\text{i}) \text{ Definition 2.5}) \\ &= F(\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H)) \circ \Phi_{H,H} \;\; (\text{naturality of } \Phi) \\ &= F(\varepsilon_H \otimes H) \circ \Phi_{H,H} \;\; ((\text{ii-1}) \text{ Definition 3.1}) \\ &= \Phi_{K,H} \circ (F(\varepsilon_H) \boxtimes F(H)) \;\; (\text{naturality of } \Phi) \\ &= \Phi_{K,H} \circ ((\Phi_0 \circ \varepsilon_{F(H)}) \boxtimes F(H)) \;\; (\text{definition of } \varepsilon_{F(H)}) \\ &= \varepsilon_{F(H)} \boxtimes F(H) \;\; ((\text{ii}) \text{ Definition 2.5}).\end{aligned}$$

Finally, if H is cocommutative, the cocommutativity of $F(H)$ follows from (15). □

4. Galois groups for Hopf quasigroups

In this section we resume the main results of [3] and we introduce the Galois group associated to strong Galois H-objects that are progenerators.

Definition 4.1. Let D be a comonoid in C. The pair (M, ρ_M) is a right D-comodule if M is an object in C and $\rho_M : M \to M \otimes D$ is a morphism in C, called the coaction, satisfying $(M \otimes \varepsilon_D) \circ \rho_M = id_M$, $(\rho_M \otimes D) \circ \rho_M = (M \otimes \delta_D) \circ \rho_M$.

Given two right D-comodules (M, ρ_M), (N, ρ_N), a morphism in C, $f\colon M \to N$, is a morphism of right D-comodules if $(f \otimes D) \circ \rho_M = \rho_N \circ f$.

The category of right D-comodules will be denoted by $\mathsf{CMod}^{\mathsf{D}}$. In a similar way we can define the category of left D-comodules denoted by ${}^{\mathsf{D}}\mathsf{CMod}$.

Definition 4.2. Let H be a Hopf quasigroup in C and let A be a unital magma (monoid) in C with a right coaction $\rho_A : A \to A \otimes H$. We will say that $\mathbb{A} = (A, \rho_A)$ is a right H-comodule magma (monoid) if (A, ρ_A) is a right H-comodule, and the following identities

(i) $\rho_A \circ \eta_A = \eta_A \otimes \eta_H$,
(ii) $\rho_A \circ \mu_A = \mu_{A\otimes H} \circ (\rho_A \otimes \rho_A)$,

hold.

Obviously, if H is a Hopf quasigroup in C, the pair $\mathbb{H} = (H, \rho_H = \delta_H)$ is an example of right H-comodule magma. Also, if H is cocommutative and $\mathbb{A}$ is a right H-comodule magma, $\overline{\mathbb{A}} = (\overline{A}, \rho_{\overline{A}} = (A \otimes \lambda_H) \circ \rho_A)$ is a right H-comodule magma (see [3, Proposition 1.6]).

A morphism of right H-comodule magmas (monoids) $f : \mathbb{A} \to \mathbb{B}$ is a morphism $f : A \to B$ in C of unital magmas (monoids) and right H-comodules.

Proposition 4.3. *Let $F : \mathsf{C} \to \mathsf{D}$ be a strong symmetric monoidal functor and let H be a Hopf quasigroup in C. Then, if $\mathbb{A}$ is a right H-comodule magma in C, $\mathbb{F}(\mathbb{A}) = (F(A), \rho_{F(A)} = \Phi^{-1}_{A,H} \circ F(\rho_A))$ is a right $F(H)$-comodule magma in D. Moreover, if $f : \mathbb{A} \to \mathbb{B}$ is a morphism of right H-comodule magmas in B, $F(f) : \mathbb{F}(\mathbb{A}) \to \mathbb{F}(\mathbb{B})$ is a morphisms of $F(H)$-comodule magmas in D.*

Proof. The proof of this result is the same that the one we can find in [12, Lemmas 4.3, 4.4] for Hopf algebras in a symmetric monoidal setting. □

Definition 4.4. Let H be a Hopf quasigroup in C and let $\mathbb{A}$ be a right H-comodule magma in C. We will say that $\mathbb{A}$ is a Galois H-object if the following conditions hold:

(i) A is faithfully flat.
(ii) The canonical morphism $\gamma_A = (\mu_A \otimes H) \circ (A \otimes \rho_A) : A \otimes A \to A \otimes H$ is an isomorphism.

Note that, by [3, Proposition 2.4], if $\mathbb{A}$ a Galois H-object,

$$K \xrightarrow{\eta_A} A \underset{A \otimes \eta_H}{\overset{\rho_A}{\rightrightarrows}} A \otimes H$$

is an equalizer diagram.

Definition 4.5. Assume that C admits coequalizers. Let H be a Hopf quasigroup in C and let $\mathbb{A}$ be a right H-comodule magma in C. We will say that $\mathbb{A}$ is a Galois H-progenerator object if the following conditions hold:

(i) A is a progenerator in C.

(ii) The canonical morphism
$$\gamma_A = (\mu_A \otimes H) \circ (A \otimes \rho_A) : A \otimes A \to A \otimes H$$
is an isomorphism.

Definition 4.6. If $\mathbb{A}$ is a Galois H-object such that $f_A = \gamma_A^{-1} \circ (\eta_A \otimes H) : H \to A^e$ is a morphism of unital magmas, we will say that $\mathbb{A}$ is a strong Galois H-object.

Definition 4.7. Assume that C admits coequalizers. A Galois H-progenerator object $\mathbb{A}$ is strong if $f_A = \gamma_A^{-1} \circ (\eta_A \otimes H) : H \to A^e$ is a morphism of unital magmas.

Note that every progenerator is a faithfully flat object. As a consequence, if C admits coequalizers, every Galois H-progenerator object is a Galois H-object, and every strong Galois H-progenerator object is strong.

Definition 4.8. A morphism between two (strong) Galois H-(progenerator) objects is a morphism of right H-comodule magmas.

Remark 4.9. Note that if $\mathbb{A}$ is a strong Galois H-(progenerator) object and $\mathbb{B}$ is a Galois H-object isomorphic to $\mathbb{A}$ as Galois H-objects, then $\mathbb{B}$ is also a strong Galois H-(progenerator) object because, if $g : A \to B$ is the isomorphism, we have $\gamma_B \circ (g \otimes g) = (g \otimes H) \circ \gamma_A$ and it follows that $f_B = (g \otimes g) \circ f_A$. Then, f_B is a morphism of unital magmas and $\mathbb{B}$ is strong. Finally, if A is a progenerator and B is isomorphic to A, B is a progenerator.

Proposition 4.10. *If H is a faithfully flat Hopf quasigroup, $\mathbb{H}$ is a strong Galois H-object. Moreover, if C admits coequalizers and H is a finite Hopf quasigroup, $\mathbb{H}$ is a strong Galois H-progenerator object.*

Proof. The pair $\mathbb{H} = (H, \delta_H)$ is a Galois H-object because $\gamma_H = (\mu_H \otimes H) \circ (H \otimes \delta_H)$ is an isomorphism with inverse $\gamma_H^{-1} = ((\mu_H \circ (H \otimes \lambda_H)) \otimes H) \circ (H \otimes \delta_H)$. Also, $\mathbb{H}$ is strong because $f_H = (\lambda_H \otimes H) \circ \delta_H : H \to H^e$ is a morphism of unital magmas. Moreover, if H is finite, by Proposition 3.5, H is a progenerator. $\square$

By [3, Proposition 2.7] we know that if H is a cocommutative Hopf quasigroup and $\mathbb{A}$ is a Galois H-object, the right H-comodule magma $\overline{\mathbb{A}}$ is a Galois H-object. Moreover, if $\mathbb{A}$ is strong so is $\overline{\mathbb{A}}$. Then, if $\mathbb{A}$ is a (strong) Galois H-progenerator object, $\overline{\mathbb{A}}$ is a (strong) Galois H-progenerator object.

To define a suitable product of Galois objects, in what follows, we will assume that C admits equalizers. By [3, Proposition 1.5], if H is a Hopf quasigroup and $\mathbb{A}$ and $\mathbb{B}$ are right H-comodule magmas, the pairs $\mathbb{A} \otimes_1 \mathbb{B} = (A \otimes B, \rho^1_{A\otimes B} = (A \otimes c_{H,B}) \circ (\rho_A \otimes B))$, $\mathbb{A} \otimes_2 \mathbb{B} = (A \otimes B, \rho^2_{A\otimes B} = A \otimes \rho_B)$ are isomorphic as right H-comodule magmas. The object $A \bullet B$ defined by the equalizer diagram

$$A \bullet B \xrightarrow{i_{A\bullet B}} A \otimes B \ \underset{\rho^2_{A\otimes B}}{\overset{\rho^1_{A\otimes B}}{\rightrightarrows}} \ A \otimes B \otimes H, \tag{20}$$

is a unital magma where $\eta_{A\bullet B}$ and $\mu_{A\bullet B}$ are the factorizations through $i_{A\bullet B}$ of the morphisms $\eta_{A\otimes B}$ and $\mu_{A\otimes B}\circ(i_{A\bullet B}\otimes i_{A\bullet B})$, respectively. Moreover, if H is flat and the coaction $\rho_{A\bullet B}: A\bullet B\to A\bullet B\otimes H$ is the factorization of $\rho^2_{A\otimes B}\circ i_{A\bullet B}$ through $i_{A\bullet B}\otimes H$, the pair $\mathbb{A}\bullet\mathbb{B}=(A\bullet B,\rho_{A\bullet B})$ is a right H-comodule magma (see [3, Proposition 1.7]). Moreover, if $f:\mathbb{A}\to\mathbb{B}$, $g:\mathbb{T}\to\mathbb{D}$ are morphisms of right H-comodule magmas, the morphism $f\bullet g: A\bullet T\to B\bullet D$, obtained as the factorization of $(f\otimes g)\circ i_{A\bullet T}: A\bullet T\to B\otimes D$ through the equalizer $i_{B\bullet D}$, is a morphism of right H-comodule magmas between $\mathbb{A}\bullet\mathbb{T}$ and $\mathbb{B}\bullet\mathbb{D}$. Finally, if f and g are isomorphisms, so is $f\bullet g$ (see [3, Proposition 1.8]).

On the other hand, if $\mathbb{A}$, $\mathbb{B}$, $\mathbb{D}$ are right H-comodule magmas, when H is flat, by [3, Proposition 1.9], we know that $\mathbb{A}\bullet\mathbb{B}$ and $\mathbb{B}\bullet\mathbb{A}$ are isomorphic as right H-comodule magmas and, if A and D are flat, $\mathbb{A}\bullet(\mathbb{B}\bullet\mathbb{D})$ and $(\mathbb{A}\bullet\mathbb{B})\bullet\mathbb{D}$ are isomorphic as right H-comodule magmas (see [3, Proposition 1.10]. Also, if H cocommutative and $\mathbb{A}$ is a right H-comodule magma

$$A \xrightarrow{\rho_A} A\otimes H \underset{\rho^2_{A\otimes H}}{\overset{\rho^1_{A\otimes H}}{\rightrightarrows}} A\otimes H\otimes H,$$

is an equalizer diagram, and $\mathbb{A}\bullet\mathbb{H}$ and $\mathbb{A}$ are isomorphic as right H-comodule magmas (see [3, Proposition 1.11]). Moreover, $\mathbb{A}\bullet\overline{\mathbb{A}}$ is isomorphic to $\mathbb{H}$ as right H-comodules and, if $\mathbb{A}$ is strong, the previous isomorphism is a morphism of right H-comodule magmas (see [3, Proposition 2.8]).

Remark 4.11. If $\mathsf{Mag}_f(\mathsf{C},\mathsf{H})$ denotes the category whose objects are flat H-comodule magmas and whose arrows are the morphisms of H-comodule magmas, when H is cocommutative and flat, we have that $\mathsf{Mag}_f(\mathsf{C},\mathsf{H})$ is a symmetric monoidal category where the tensor product is defined by the product " $\bullet$ ", the unit is $\mathbb{H}$, the associative constraints $\mathfrak{a}_{\mathbb{A},\mathbb{B},\mathbb{D}}=n^{-1}_{A,B,D}$, where $n_{A,B,D}$ is the isomorphism between $\mathbb{A}\bullet(\mathbb{B}\bullet\mathbb{D})$ and $(\mathbb{A}\bullet\mathbb{B})\bullet\mathbb{D}$, and the right unit constraints and the left unit constraints are $\mathfrak{r}_{\mathbb{A}}=r_A$, $\mathfrak{l}_{\mathbb{A}}=r_A\circ\tau_{H,A}$ respectively, where r_A is the isomorphism between $\mathbb{A}\bullet\mathbb{H}$ and $\mathbb{A}$ and $\tau_{H,A}$ is the one between $\mathbb{H}\bullet\mathbb{A}$ and $\mathbb{A}\bullet\mathbb{H}$ (see [3, Proposition 1.13] for details).

Proposition 4.12. *Let H be a cocommutative faithfully flat Hopf quasigroup in C. The following assertions hold*:

(i) *If C admits equalizers and $\mathbb{A}$, $\mathbb{B}$ are Galois H-objects, so is $\mathbb{A}\bullet\mathbb{B}$.*
(ii) *If C admits equalizers and $\mathbb{A}$, $\mathbb{B}$ are strong Galois H-objects, so is $\mathbb{A}\bullet\mathbb{B}$.*
(iii) *If C admits equalizers and coequalizers H is finite and $\mathbb{A}$, $\mathbb{B}$ are Galois H-progenerator objects, so is $\mathbb{A}\bullet\mathbb{B}$. As a consequence, if $\mathbb{A}$ and $\mathbb{B}$ are strong Galois H-progenerator objects so is $\mathbb{A}\bullet\mathbb{B}$.*

Proof. The proof for (i) and (ii) is the one given in [3, Proposition 2.6]. To get (iii), note that in the proof of (i) we show that there exists an isomorphism $g: A\otimes B\otimes H\to A\otimes B\otimes A\bullet B$. Then, using that $A\otimes B$ is a progenerator, we obtain that $H \cong A \bullet B$ and, as a consequence, $A \bullet B$ is a progenerator because, by Proposition 4.10, H is a progenerator. $\square$

As a consequence of the previous proposition and [3, Theorem 2.10], we have the following: Assume that C admits equalizers and let H be a cocommutative faithfully flat Hopf quasigroup in C. Let $Gal(H)$ be the set of isomorphism classes of Galois H-objects, and for a Galois H-object $\mathbb{A}$ denote its class in $Gal(H)$ by $[\mathbb{A}]$. With the product defined by

$$[\mathbb{A}].[\mathbb{B}] = [\mathbb{A} \bullet \mathbb{B}] \tag{21}$$

and unit $[\mathbb{H}]$, $Gal(H)$ is a commutative monoid. In addition, if H is finite, C admits coequalizers, and $Gal^p(H)$ denotes the set of isomorphism classes of Galois H-progenerator objects, we obtain that $Gal^p(H)$ is a commutative monoid.

If we denote by $Gal^s(H)$ the set of isomorphism classes of strong Galois H-objects, with the product defined in (21) for Galois H-objects, $Gal^s(H)$ is a commutative group because by (ii) of Proposition 4.12 the product of strong Galois H-objects is a strong Galois H-object, by Proposition 4.10 we know that $\mathbb{H}$ is a strong Galois H-object and by [3, Propositions 2.7, 2.8] the inverse of $[\mathbb{A}]$ in $Gal^s(H)$ is $[\overline{\mathbb{A}}]$. If in addition, H is finite, C admits coequalizers, and $Gal^{sp}(H)$ denotes the set of isomorphism classes of strong Galois H-progenerator objects, we obtain that $Gal^{sp}(H)$ is a commutative group. Note that $Gal(H)$ and $Gal^p(H)$ are submonoids of $Gal^s(H)$ and $Gal^{sp}(H)$, respectively, and on the other hand, $Gal^{sp}(H)$ is a subgroup of $Gal^s(H)$.

Proposition 4.13. *Assume that* C *admits coequalizers and let* D *be symmetric monoidal category with coequalizers. Let* $F : \mathsf{C} \to \mathsf{D}$ *be a strong symmetric monoidal functor such that preserves coequalizers. Let* H *be a Hopf quasigroup in* C. *The following assertions hold*:

(i) *If* $\mathbb{A}$ *is a Galois* H*-progenerator object in* C *so is* $\mathbb{F}(\mathbb{A})$ *in* D.
(ii) *If* $\mathbb{A}$ *is a strong Galois* H*-progenerator object in* C *so is* $\mathbb{F}(\mathbb{A})$ *in* D.

Proof. The proof for (i) follows from Proposition 4.3 and from the following fact: The canonical morphism $\gamma_{F(A)}$ is an isomorphism with inverse $\gamma_{F(A)}^{-1} = \Phi_{A,A}^{-1} \circ F(\gamma_A^{-1}) \circ \Phi_{A,H}$ (the proof is similar to the one given in [12, Lemma 4.3] for Hopf algebras).

On the other hand, by (11) and the naturality of ϕ, we obtain that $f_{F(A)} = F(f_A)$ and then $f_{F(A)}$ is a morphism of unital magmas because, as was pointed in Section 2, every monoidal functor preserves the condition of morphism of unital magmas. Therefore (ii) holds. $\square$

Proposition 4.14. *Assume that* C *admits equalizers and coequalizers and let* D *be a symmetric monoidal category with equalizers and coequalizers. Let* $F : \mathsf{C} \to \mathsf{D}$ *be a strong symmetric monoidal functor preserving equalizers and* *coequalizers. Let H be a cocommutative finite Hopf quasigroup in C. The map*

$$Gal^p(F) : Gal^p(H) \to Gal^p(F(H))$$

defined by

$$Gal^p(F)([\mathbb{A}]) = [\mathbb{F}(\mathbb{A})]$$

is a monoid morphism. Moreover, the restriction of the previous map to $Gal^{sp}(H)$, induces a group morphism

$$Gal^{sp}(F) : Gal^{sp}(H) \to Gal^{sp}(F(H)).$$

Proof. The proof follows the same pattern of the one developed for Hopf algebras in [12, Proposition 4.5]. The map $Gal^p(F)$ is well-defined by (i) of Proposition 4.13. To prove that $Gal^p(F)$ is a morphism of groups we only need to see that if $\mathbb{A}$ and $\mathbb{B}$ are Galois H-progenerator objects then $\mathbb{F}(\mathbb{A}) \bullet \mathbb{F}(\mathbb{B}) \cong \mathbb{F}(\mathbb{A} \bullet \mathbb{B})$ as Galois $F(H)$-progenerator objects.

Let $\mathbb{A}$ and $\mathbb{B}$ be Galois H-progenerator objects. The morphism

$$\Phi_{A,B}^{-1} \circ F(i_{A\bullet B}) : F(A \bullet B) \to F(A) \boxtimes F(B)$$

satisfies:

$$\begin{aligned} &\Phi_{A\otimes B,H} \circ (\Phi_{A,B} \boxtimes F(H)) \circ \rho^1_{F(A)\boxtimes F(B)} \circ \Phi_{A,B}^{-1} \circ F(i_{A\bullet B}) \\ = {}&\Phi_{A\otimes B,H} \circ (\Phi_{A,B} \boxtimes F(H)) \circ \rho^2_{F(A)\boxtimes F(B)} \circ \Phi_{A,B}^{-1} \circ F(i_{A\bullet B}). \end{aligned}$$

Therefore, there exists a unique morphism

$$p_{AB} : F(A \bullet B) \to F(A) \bullet F(B)$$

satisfying

$$i_{F(A)\bullet F(B)} \circ p_{AB} = \Phi_{A,B}^{-1} \circ F(i_{A\bullet B}) \tag{22}$$

and such that it is a morphism of right $F(H)$-comodule magmas, i.e., a morphism of Galois H-progenerator objects.

On the other hand, if F preserves equalizers

$$\begin{tikzcd} F(A \bullet B) \arrow[r, "F(i_{A\bullet B})"] & F(A \otimes B) \arrow[r, shift left, "F(\rho^1_{A\otimes B})"] \arrow[r, shift right, "F(\rho^2_{A\otimes B})"'] & F(A \otimes B \otimes H), \end{tikzcd}$$

is an equalizer diagram. Moreover,

$$\begin{aligned} &(\Phi_{A,B}^{-1} \boxtimes F(H)) \circ \Phi_{A\otimes B,H}^{-1} \circ F(\rho^1_{A\otimes B}) \circ \Phi_{A,B} \circ i_{F(A)\bullet F(B)} \\ = {}&(\Phi_{A,B}^{-1} \boxtimes F(H)) \circ \Phi_{A\otimes B,H}^{-1} \circ F(\rho^2_{A\otimes B}) \circ \Phi_{A,B} \circ i_{F(A)\bullet F(B)} \end{aligned}$$

holds. Thus, there exists a unique morphism

$$q_{AB} : F(A) \bullet F(B) \to F(A \bullet B)$$

satisfying

$$\Phi_{A,B} \circ i_{F(A)\bullet F(B)} = F(i_{A\bullet B}) \circ q_{AB}. \tag{23}$$

Then,

$$F(i_{A\bullet B}) \circ q_{A,B} \circ p_{A,B} \overset{(23)}{=} \Phi_{A,B} \circ i_{F(A)\bullet F(B)} \circ p_{A,B} \overset{(22)}{=} \Phi_{A,B} \circ \Phi^{-1}_{A,B} \circ F(i_{A\bullet B}).$$

Therefore, using that $F(i_{A\bullet B})$ is a monomorphism, we have that $q_{A,B} \circ p_{A,B} = id_{F(A\bullet B)}$. Similarly, we obtain the identity $p_{A,B} \circ q_{A,B} = id_{F(A)\bullet F(B)}$ and this implies that $p_{A,B}$ is an isomorphism of Galois H-progenerator objects. As a consequence, $Gal^{p}(F)$ and $Gal^{sp}(F)$ are group morphisms. $\square$

Example 4.15. Let Set be the category of sets. Then, Set with $\otimes = \times$ and $K = \{*\}$ is a symmetric monoidal category with equalizers and coequalizers. Let R be a commutative ring. The free R-module functor

$$R[\ \ ] : \mathsf{Set} \rightarrow {}_{\mathsf{R}}\mathsf{Mod},$$

defined by

$$R[X] = \bigoplus_{x\in X} Rx$$

on a set X and in the obvious way on maps, is an example of strong symmetric monoidal functor. Then, the free module construction turns set-theoretic products into tensor products and the twist on sets into the twist on the category of left R-modules. Also it preserves coequalizers because it is left adjoint to the forgetful functor and it is easy to show that $R[\ \ ]$ preserves equalizers. Note that a Hopf quasigroup in Set is an IP loop L. Then, the loop algebra $R[L]$, defined in Example 3.3, has the Hopf quasigroup structure induced by the free R-module functor (see Proposition 3.6).

If $\alpha : R \to S$ is a homomorphism of commutative rings, then the restriction of scalars is a monoidal functor $G : {}_{\mathsf{S}}\mathsf{Mod} \to {}_{\mathsf{R}}\mathsf{Mod}$. The extension of scalars (or induction functor) $F : {}_{\mathsf{R}}\mathsf{Mod} \to {}_{\mathsf{S}}\mathsf{Mod}$, defined by

$$F((M, \varphi_M)) = (S \otimes_R M, \varphi_{S\otimes_R M})$$

on objects and by $F(f) = S\otimes_R f$ on morphisms, is a strong symmetric monoidal functor preserving coequalizers because it is left adjoint to the functor G. Moreover if S is R-flat, F is exact and then preserves equalizers. For example, if T is a multiplicative closed set of R, there exists a ring morphism $i_T : R \to T^{-1}R$ and $S = T^{-1}R$ is a commutative ring that is flat as R-module. In this case, for any left R-module $F((M, \varphi_M)) = T^{-1}M$, i.e., the localization of (M, φ_M) by T.

Finally, if $\mathbb{F}$ is a field and S is a $\mathbb{F}$-algebra, S contains $\mathbb{F}$ as subring and it is flat as $\mathbb{F}$-module. Then these monomorphisms of rings provide examples of strong symmetric monoidal functors preserving equalizers and coequalizers.

Definition 4.16. Let H be a faithfully flat Hopf quasigroup in C. If $\mathbb{A}$ is a Galois H-object, we will say that $\mathbb{A}$ has a normal basis if (A, ρ_A) is isomorphic to (H, δ_H) as right H-comodules. We denote by n_A the H-comodule isomorphism between A and H.

If C admits equalizers and H is cocommutative, the set of isomorphism classes of Galois H-objects with normal basis, denoted by $N(H)$, is a submonoid of $Gal(H)$ because $\mathbb{H} = (H, \delta_H)$ is a Galois H-object with normal basis and if $\mathbb{A}$, $\mathbb{B}$ are Galois H-objects with normal basis and associated isomorphisms n_A, n_B, respectively, then $\mathbb{A} \bullet \mathbb{B}$ is a Galois H-object with normal basis and associated H-comodule isomorphism $n_{A\bullet B} = r_H \circ (n_A \bullet n_B)$ where $n_A \bullet n_B$ is defined in [3, Proposition 1.8]) and r_H is the isomorphism between $\mathbb{H}\bullet\mathbb{H}$ and $\mathbb{H}$. Moreover, for a strong Galois H-object with normal basis $\mathbb{A}$, with associated isomorphism n_A, we have that $\overline{\mathbb{A}} = (\overline{A}, \rho_{\overline{A}})$ is also a strong Galois H-object with normal basis, where $n_{\overline{A}} = \lambda_H \circ n_A$, and then, if we denote by $N^s(H)$ the set of isomorphism classes of strong Galois H-objects with normal basis, $N^s(H)$ is a subgroup of $Gal^s(H)$.

Furthermore, if C admits coequalizers and H is finite, when $\mathbb{A}$ has a normal basis we obtain that $\mathbb{A}$ is a Galois H-progenerator object because A is isomorphic to H. Therefore, under these conditions $N(H) = N^p(H)$ where $N^p(H)$ denotes the set of isomorphism classes of Galois H-progenerator objects with normal basis. Similarly, $N^s(H) = N^{sp}(H)$, where $N^{sp}(H)$ denotes the set of isomorphism classes of strong Galois H-progenerator objects with normal basis.

Proposition 4.17. *Assume that C admits equalizers and coequalizers and let D be a symmetric monoidal category with equalizers and coequalizers. Let $F : \mathsf{C} \to \mathsf{D}$ be a strong symmetric monoidal functor preserving equalizers and coequalizers. Let H be a finite cocommutative Hopf quasigroup in C. The map*

$$N^p(F) : N^p(H) \to N^p(F(H))$$

defined by

$$N^p(F)([\mathbb{A}]) = [\mathbb{F}(\mathbb{A})]$$

is a monoid morphism. Moreover, the restriction of the previous map to $N^{sp}(H)$, induces a group morphism

$$N^{sp}(F) : N^{sp}(H) \to N^{sp}(F(H)).$$

Proof. The proof follows from Proposition 4.14 and from the following fact: If $\mathbb{A}$ is a right H-comodule magma isomorphic to $\mathbb{H}$ as right H-comodules, the image of $\mathbb{A}$ by F is isomorphic to $\mathbb{F}(\mathbb{H})$ as right H-comodules. □

Remark 4.18. Note that, in the conditions of the previous proposition,

$$\begin{tikzcd} N^{sp}(H) \arrow[r, "N^{sp}(F)"] \arrow[d, "i_H^{sp}"'] & N^{sp}(F(H)) \arrow[d, "i_{F(H)}^{sp}"] \\ Gal^{sp}(H) \arrow[r, "Gal^{sp}(F)"'] & Gal^{sp}(F(H)) \end{tikzcd}$$

is a commutative diagram of commutative groups where i_H^{sp} and $i_{F(H)}^{sp}$ denote the obvious monomorphisms. Similarly, we have a commutative diagram of monoids when we work with not strong Galois H-progenerator objects:

$$\begin{tikzcd} N^p(H) \arrow[r, "N^p(F)"] \arrow[d, "i_H^p"'] & N^p(F(H)) \arrow[d, "i^p_{F(H)}"] \\ Gal^p(H) \arrow[r, "Gal^p(F)"'] & Gal^p(F(H)) \end{tikzcd}$$

5. Exact sequences for Galois groups

In this section we will construct exact sequences of groups associated to a finite cocommutative Hopf quasigroup in a symmetric monoidal category C with equalizers and coequalizers. Under these conditions H is a progenerator object and then all object isomorphic to H is a progenerator object.

We define the symmetric monoidal category $\mathsf{Gal(C,H)}$ as the one whose objects are Galois H-objects, whose morphisms are morphisms of right H-comodule magmas, the tensor product $\bullet$ is the one defined by the equalizer diagram (20), the unit object is $\mathbb{H}$, and, finally the associative and unit constraints are the ones defined for $\mathsf{Mag}_f\mathsf{(C,H)}$ (see Remark 4.11). Similarly, with the same morphisms, tensor product, unit object and associative and unit constraints we can define new symmetric monoidal categories: $\mathsf{Gal^s(C,H)}$ whose objects are strong Galois H-objects, $\mathsf{Gal^p(C,H)}$ whose objects are Galois H-progenerator objects, $\mathsf{Gal^{sp}(C,H)}$ whose objects are strong Galois H-progeneretor objects, $\mathsf{N(C,H)}$ whose objects are Galois H-objects with normal basis, and $\mathsf{N^s(C,H)}$ whose objects are strong Galois H-objects with normal basis. Note that, in this case, $\mathsf{N(C,H)} = \mathsf{N^p(C,H)}$, where $\mathsf{N^p(C,H)}$ is the symmetric monoidal category whose objects are Galois H-progenerator objects with normal basis, and similarly $\mathsf{N^s(C,H)} = \mathsf{N^{sp}(C,H)}$.

Then, we have a commutative diagram

$$\begin{tikzcd} \mathsf{N^p(C,H)} = \mathsf{N(C,H)} \arrow[rr, "J"] \arrow[dr, "J^p"] & & \mathsf{Gal(C,H)} \\ & \mathsf{Gal^p(C,H)} \arrow[ur, "I_p"] & \\ \mathsf{N^{sp}(C,H)} = \mathsf{N^s(C,H)} \arrow[uu, "L_s"] \arrow[ur, "J_s^p"] \arrow[rr, "J_s^s"'] \arrow[dr, "J_s^{sp}"'] & & \mathsf{Gal^s(C,H)} \arrow[uu, "I_s"'] \\ & \mathsf{Gal^{sp}(C,H)} \arrow[uu, "I_{sp}^p"] \arrow[ur, "I_{sp}^s"'] & \end{tikzcd}$$

where I_{sp}^p, I_{sp}^s, I_p, J, J^p, J_s^p, J_s^{sp}, J_s^s and L_s are inclusion functors.

Let E be a category where the isomorphisms classes of E form a set. Following [5] we will say that E is a category with product if there exists a functor $\odot_{\mathsf{E}} : \mathsf{E} \times \mathsf{E} \to \mathsf{E}$ and natural isomorphisms $\odot_{\mathsf{E}} \circ (Id_{\mathsf{E}} \times \odot_{\mathsf{E}}) \cong \odot_{\mathsf{E}} \circ (\odot_{\mathsf{E}} \times Id_{\mathsf{E}}) : \mathsf{E} \times \mathsf{E} \times \mathsf{E} \to \mathsf{E}$ and $\odot_{\mathsf{E}} \circ \tau \cong \odot_{\mathsf{E}} : \mathsf{E} \times \mathsf{E} \to \mathsf{E}$ where τ is the twist. The Grothendieck group $K_0\mathsf{E}$ is the abelian group generated by the isomorphisms classes $[X]$ of objects of E modulo the relations $[X \odot_{\mathsf{E}} Y] = [X][Y]$. Note that for X, Y objects in E, $[X] = [Y]$ in $K_0\mathsf{E}$ if and only if there exists Z in E such that $X \odot_{\mathsf{E}} Z \cong Y \odot_{\mathsf{E}} Z$ in E. Also it is easy to show that any element in $K_0\mathsf{E}$ is equal to $[X][Y]^{-1}$ for some objects X, Y in E.

On the other hand, if E and T are categories with product and $\Gamma : \mathsf{E} \to \mathsf{T}$ is a product preserving functor, i.e., for all objects X, Y in E there are natural isomorphisms $\Gamma(X \odot_{\mathsf{E}} Y) \cong \Gamma(X) \odot_{\mathsf{T}} \Gamma(Y)$, we have a group morphism $K_0\Gamma : K_0\mathsf{E} \to K_0\mathsf{T}$.

If we apply this construction to the categories of Galois objects associated to a finite cocommutative Hopf quasigroup (all of them categories with product $\odot = \bullet$) we have that

$$K_0\mathsf{Gal}^{\mathsf{s}}(\mathsf{C},\mathsf{H}) = Gal^s(H), \quad K_0\mathsf{Gal}^{\mathsf{sp}}(\mathsf{C},\mathsf{H}) = Gal^{sp}(H),$$
$$K_0\mathsf{N}^{\mathsf{s}}(\mathsf{C},\mathsf{H}) = N^s(H) = N^{sp}(H).$$

Also, using that the inclusion functors I_{sp}^p, I_{sp}^s, I_p, J, J^p, J_s^p, J_s^{sp}, J_s^s and L_s preserve the product, we have group morphisms

$$K_0I_{sp}^p : Gal^{sp}(H) \to K_0\mathsf{Gal}^{\mathsf{p}}(\mathsf{C},\mathsf{H}), \quad K_0I_{sp}^s : Gal^{sp}(H) \to Gal^s(H),$$
$$K_0I_p : K_0\mathsf{Gal}^{\mathsf{p}}(\mathsf{C},\mathsf{H}) \to K_0\mathsf{Gal}(\mathsf{C},\mathsf{H}), \quad K_0J : K_0\mathsf{N}^{\mathsf{p}}(\mathsf{C},\mathsf{H}) \to K_0\mathsf{Gal}(\mathsf{C},\mathsf{H}),$$
$$K_0J^p : K_0\mathsf{N}^{\mathsf{p}}(\mathsf{C},\mathsf{H}) \to K_0\mathsf{Gal}^{\mathsf{p}}(\mathsf{C},\mathsf{H}), \quad K_0J_s^p : N^{sp}(H) \to K_0\mathsf{Gal}^{\mathsf{p}}(\mathsf{C},\mathsf{H}),$$
$$K_0J_s^{sp} : N^{sp}(H) \to Gal^{sp}(H), \quad K_0J_s^s : N^{sp}(H) \to Gal^s(H),$$
$$K_0L_s : N^{sp}(H) \to K_0\mathsf{N}(\mathsf{C},\mathsf{H}).$$

Proposition 5.1. *The following assertions hold*:

(i) $Ker(K_0I_{sp}^p) \subset K_0\mathsf{N}^{\mathsf{p}}(\mathsf{C},\mathsf{H})$.
(ii) $Ker(K_0I_p) \subset K_0\mathsf{N}^{\mathsf{p}}(\mathsf{C},\mathsf{H})$.
(iii) $[\mathbb{B}] \in Ker(K_0J)$ *if and only if there exists* $\mathbb{D}$ *in* $\mathsf{Gal}(\mathsf{C}, H)$ *such that* $\mathbb{B} \bullet \mathbb{D} \cong \mathbb{D}$ *as right* H*-comodule magmas.*
(iv) $[\mathbb{B}] \in Ker(K_0J^p)$ *if and only if there exists* $\mathbb{D}$ *in* $\mathsf{Gal}^{\mathsf{p}}(\mathsf{C}, H)$ *such that* $\mathbb{B} \bullet \mathbb{D} \cong \mathbb{D}$ *as right* H*-comodule magmas.*
(v) $[\mathbb{B}] \in Ker(K_0J_s^p)$ *if and only if there exists* $\mathbb{D}$ *in* $\mathsf{Gal}^{\mathsf{p}}(\mathsf{C}, H)$ *such that* $\mathbb{B} \bullet \mathbb{D} \cong \mathbb{D}$ *as right* H*-comodule magmas.*
(vi) $K_0I_{sp}^s$, $K_0J_s^{sp}$, $K_0J_s^s$ *and* K_0L_s *are monomorphisms.*

Proof. First we prove (i). Let $[\mathbb{B}]$ be in $Ker(K_0I_{sp}^p)$. Then $[\mathbb{B}] = [\mathbb{H}]$ in $K_0\mathsf{Gal}^{\mathsf{p}}(\mathsf{C},\mathsf{H})$. Thus, there exists $\mathbb{D}$ in $\mathsf{Gal}^{\mathsf{p}}(\mathsf{C},\mathsf{H})$ such that $\mathbb{B} \bullet \mathbb{D} \cong \mathbb{H} \bullet \mathbb{D}$ as right H-comodule magmas. This implies that $\mathbb{B} \bullet \mathbb{D} \cong \mathbb{D}$ as right H-comodule magmas and, as a consequence, $\mathbb{B} \cong \mathbb{H}$ as right H-comodules because $\mathbb{D}$ is not strong. Therefore $\mathbb{B}$ is in $\mathsf{N}^{\mathsf{p}}(\mathsf{C},\mathsf{H})$ and $[\mathbb{B}]$ belongs to $K_0\mathsf{N}^{\mathsf{p}}(\mathsf{C},\mathsf{H})$. The proofs for (ii)-(vi) are similar and we leave the details to the reader. Finally, note that if is $[\mathbb{B}]$ in $Ker(K_0 J_{sp}^s)$, $\mathbb{B} \cong \mathbb{H}$ as right H-comodules magmas, and then $[\mathbb{B}]$ is the unit of $Gal_{\mathsf{C}}^{sp}(H)$. Similarly we obtain that $K_0 J_s^{sp}$, $K_0 J_s^s$ and $K_0 L_s$ are monomorphisms. □

Remark 5.2. Note that in (iii), (iv) and (v) of the previous proposition the isomorphism of right H-comodule magmas $\mathbb{B} \bullet \mathbb{D} \cong \mathbb{D}$ only implies that $\mathbb{B} \cong \mathbb{H}$ as right H-comodules because $\mathbb{D}$ is not strong.

If E is a category with product $\odot_{\mathsf{E}}$, it is possible to define a new category with product denoted by $\mathsf{L}(\mathsf{E})$ and called the loop category of E. The objects of this category are pairs (X, α) with X an object in E and α an E-automorphism of X. A morphism $f : (X, \alpha) \to (Y, \beta)$ in $\mathsf{L}(\mathsf{E})$ is a morphism $f : X \to Y$ in E such that $\beta \circ f = f \circ \alpha$. The product on $\mathsf{L}(\mathsf{E})$ is defined with the product of E in the two components of each object and as in E on morphisms. The Whitehead group of E is defined as the Grothendieck group of $\mathsf{L}(\mathsf{E})$ modulo the subgroup generated by elements of the form $[(X, \alpha \circ \beta)] = [(X, \alpha)][(X, \beta)]$. In the following we will denote this group by $K_1\mathsf{E}$. As in the case of the Grothendieck group, it is easy to show that if $\Gamma : \mathsf{E} \to \mathsf{T}$ is a product preserving functor we have a group morphism $K_1\Gamma : K_1\mathsf{E} \to K_1\mathsf{T}$.

Let $\Gamma : \mathsf{E} \to \mathsf{T}$ be a product preserving functor. We will say that Γ is cofinal if, for all Z objects in T there exist an object Y in T and an object X in E such that $\Gamma(X) \cong Z \odot_{\mathsf{T}} Y$. For these kind of functors we define a new category with product, denoted by $\Phi(\Gamma)$, whose objects are triples (X, α, Y) with X, Y objects in E and $\alpha : \Gamma(X) \to \Gamma(Y)$ an isomorphism in T. The morphisms of $\Phi(\Gamma)$ are pairs $(f, g) : (X, \alpha, Y) \to (X', \alpha', Y')$ where $f : X \to X'$, $g : Y \to Y'$ are morphisms in E such that $\Gamma(g) \circ \alpha = \alpha' \circ \Gamma(f)$. The product on objects is defined componentwise using the product of E and in a similar way for morphisms. The abelian group $K_1\Phi(\Gamma)$ is defined as $K_0\Phi(\Gamma)$ modulo the subgroup generated by elements of the form $[(X, \beta \circ \alpha, Z)][(X, \alpha, Y)]^{-1}[(Y, \beta, Z)]^{-1}$. With these definitions we have that there exists, for any cofinal functor Γ, an exact sequence of abelian groups (see [5])

$$K_1\mathsf{E} \xrightarrow{K_1\Gamma} K_1\mathsf{T} \xrightarrow{d} K_1\Phi(\Gamma) \xrightarrow{l} K_0\mathsf{E} \xrightarrow{K_0\Gamma} K_0\mathsf{T}$$

with $l([(X, \alpha, Y)]) = [Y][X]^{-1}$ and $d([(Z, \omega)]) = [(X, \beta, X)]$, where the object X and the morphism β come from $[(Z, \omega)] = [(Z \odot_{\mathsf{T}} Y, \omega \odot_{\mathsf{T}} id_Y)] = [(\Gamma(X), \beta)]$ in $K_1\mathsf{T}$.

Let E be a category with product. A full subcategory with product E_0 is called a cofinal subcategory of E if the inclusion functor I_{E_0} is cofinal. Then $K_1 I_{\mathsf{E}_0}$ is an isomorphism, $K_0 I_{\mathsf{E}_0}$ is a monomorphism, and $K_1\Phi(I_{\mathsf{E}_0})$ is trivial. Moreover, if there exists an object X in E such that the full subcategory $\{X\}$ is cofinal we have that $K_1\mathsf{E} \cong Aut_{\mathsf{E}}(X)$ (see [5]).

Proposition 5.3. *The functors I_{sp}^{s}, J_{s}^{sp}, and J_{s}^{s} are cofinal. Also $\{\mathbb{H}\}$ is a cofinal subcategory of* $\mathsf{Gal^s(C,H)}$, $\mathsf{Gal^{sp}(C,H)}$, $\mathsf{N^{sp}(C,H)}$. *Therefore,*

$$K_1\mathsf{Gal^s(C,H)} \cong K_1\mathsf{Gal^{sp}(C,H)} \cong K_1\mathsf{N^{sp}(C,H)} \cong Aut(\mathbb{H})$$

where $Aut(\mathbb{H})$ is the group of automorphisms of right H-comodule magmas of $\mathbb{H}$.

Proof. Note that for every object $\mathbb{B}$ in the categories $\mathsf{Gal^s(C,H)}$, $\mathsf{Gal^{sp}(C,H)}$, $\mathsf{N^{sp}(C,H)}$, we have an isomorphism of right H-comodule magmas $\mathbb{B}\bullet\overline{\mathbb{B}} \cong \mathbb{H}$. Therefore, I_{sp}^{s}, J_{s}^{sp}, and J_{s}^{s} are cofinal and $\{\mathbb{H}\}$ is a cofinal subcategory of $\mathsf{Gal^s(C,H)}$, $\mathsf{Gal^{sp}}(\mathsf{C},H)$, $\mathsf{N^{sp}(C,H)}$. Thus by the results of the previous paragraph we obtain the isomorphisms for the Whitehead groups. $\square$

Proposition 5.4. *Assume that* C *admits equalizers and coequalizers and let* D *be a symmetric monoidal category with equalizers and coequalizers. Let $F : \mathsf{C} \to \mathsf{D}$ be a strong symmetric monoidal functor preserving equalizers and coequalizers. Let H be a cocommutative finite Hopf quasigroup in* C. *The functors*

$$G(F) : \mathsf{Gal^{sp}(C,H)} \to \mathsf{Gal^{sp}(D,F(H))}, \quad N(F) : \mathsf{N^{sp}(C,H)} \to \mathsf{N^{sp}(D,F(H))},$$

defined by

$$G(F)(\mathbb{A}) = \mathbb{F}(\mathbb{A}), \;\; N(F)(\mathbb{A}) = \mathbb{F}(\mathbb{A})$$

on objects and in the obvious way for morphisms, are cofinal. Therefore there exist two exact sequences of abelian groups:

$$Aut(\mathbb{H}) \to Aut(\mathbb{F}(\mathbb{H})) \to K_1\Phi(G(F)) \to Gal^{sp}(H) \overset{G(F)}{\to} Gal^{sp}(F(H)),$$

$$Aut(\mathbb{H}) \to Aut(\mathbb{F}(\mathbb{H})) \to K_1\Phi(N(F)) \to N^{sp}(H) \overset{N(F)}{\to} N^{sp}(F(H)).$$

Proof. The functors $G(F)$ and are well defined and they are cofinal because for every object $\mathbb{B}$ in the categories $\mathsf{Gal^{sp}}(\mathsf{C},F(H))$, $\mathsf{N^{sp}}(\mathsf{C},F(H))$, we have that $\mathbb{B}\bullet\overline{\mathbb{B}} \cong \mathbb{F}(\mathbb{H})$ as right $F(H)$-comodule magmas. Thus, applying the general theory of exact sequences associated to cofinal functors we obtain the result. $\square$

As was proved in [3], the group $Aut(\mathbb{H})$ admits a good explanation in terms of the group of grouplike elements of the dual algebra of H when H is finite. If H is a finite cocommutative Hopf quasigroup, H^* is a finite commutative Hopf coquasigroup and, if $G(H^*)$ (the set of grouplike elements of H^*) denotes the set of morphisms $h : K \to H^*$ such that $\delta_{H^*} \circ h = h \otimes h$ and $\varepsilon_{H^*} \circ h = id_K$, $G(H^*)$ with the convolution product $h * g = \mu_{H^*} \circ (h \otimes g)$ is a commutative group, called the group of grouplike morphisms of H^*. Note that the unit element of $G(H^*)$ is η_{H^*} and the inverse of $h \in G(H^*)$ is $h^{-1} = \lambda_{H^*} \circ h$. Under this conditions we have that $G(H^*)$ and $Aut(\mathbb{H})$ are isomorphic (see [3]). Then we have the following corollary.

Corollary 5.5. *In the conditions of Proposition* 5.4 *there exists two exact sequences of abelian groups*:

$$G(H^*) \to G(F(H)^*) \to K_1\Phi(G(F)) \to Gal^{sp}(H) \stackrel{G(F)}{\to} Gal^{sp}(F(H)),$$

$$G(H^*) \to G(F(H)^*) \to K_1\Phi(N(F)) \to N^{sp}(H) \stackrel{N(F)}{\to} N^{sp}(F(H)).$$

6. Exact sequences for invertible modules

In the last section of this paper we will study the exact sequences associated to invertible (quasi)modules and associated to Hopf quasigroups in a symmetric monoidal category C with equalizers and coequalizers.

Definition 6.1. Let H be a Hopf quasigroup in C. We say that (M, φ_M) is a left H-quasimodule if M is an object in C and $\varphi_M : H \otimes M \to M$ is a morphism in C (called the action) satisfying

$$\varphi_M \circ (\eta_H \otimes M) = id_M, \tag{24}$$

$$\begin{aligned} &\varphi_M \circ (H \otimes \varphi_M) \circ (H \otimes \lambda_H \otimes M) \circ (\delta_H \otimes M) \\ &= \varepsilon_H \otimes M = \varphi_M \circ (\lambda_H \otimes \varphi_M) \circ (\delta_H \otimes M). \end{aligned} \tag{25}$$

This definition was introduced by Brzeziński and Jiao in [8], but the involved equalities appeared previously in the definition of Hopf module that we can find in [7] and in [14].

Given two left H-quasimodules (M, φ_M) and (N, φ_N), $f : M \to N$ is a morphism of left H-quasimodules if it is a morphism of left H-modules for the unital magma H, i.e., $\varphi_N \circ (H \otimes f) = f \circ \varphi_M$ holds. We denote the category of left H-quasimodules by ${}_{\mathsf{H}}\mathsf{QMod}$.

If (M, φ_M) and (N, φ_N) are left H-quasimodules, the tensor product $M \otimes N$ is a left H-quasimodule with the diagonal action

$$\varphi_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N). \tag{26}$$

The tensor product defined by the diagonal action (26) makes the category of left H-quasimodules into a monoidal category $({}_{\mathsf{H}}\mathsf{QMod}, \otimes, K)$ and in a symmetric monoidal category when H is cocommutative. Therefore, if H is cocommutative, ${}_{\mathsf{H}}\mathsf{QMod}$ is a category with product. Replacing (25) by the equality

$$\varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M), \tag{27}$$

we obtain the definition of left H-module. Under these conditions, (25) holds trivially. Note that (H, μ_H) is not an H-module but an H-quasimodule. The morphism between left H-modules is defined as for H-quasimodules and we denote the category of left H-modules by ${}_{\mathsf{H}}\mathsf{Mod}$. As in the case of left H-quasicomodules, ${}_{\mathsf{H}}\mathsf{Mod}$ with the tensor product defined by the diagonal structure is monoidal with unit object K and also is a symmetric category, and therefore a category with product, when H is cocommutative.

In the Hopf quasigroup setting the notion of left H-comodule is exactly the same as for ordinary Hopf algebras since it only depends on the coalgebra structure of H. Then, we will denote a left H-comodule by (M, ϱ_M) where M is an object in C and $\varrho_M : M \to H \otimes M$ is a morphism in C (called the coaction) satisfying the comodule conditions:

$$(\varepsilon_H \otimes M) \circ \varrho_M = id_M, \tag{28}$$

$$(H \otimes \varrho_M) \circ \varrho_M = (\delta_H \otimes M) \circ \varrho_M. \tag{29}$$

Given two left H-comodules (M, ϱ_M) and (N, ϱ_N), $f : M \to N$ is a morphism of left H-comodules if

$$\varrho_N \circ f = (H \otimes f) \circ \varrho_M. \tag{30}$$

In the following, we denote the category of left H-comodules by ${}^{\mathsf{H}}\mathsf{Comod}$. For two left H-comodules (M, ϱ_M) and (N, ϱ_N), the tensor product $M \otimes N$ is a left H-comodule with the codiagonal coaction

$$\varrho_{M\otimes N} = (\mu_H \otimes M \otimes N) \circ (H \otimes c_{M,H} \otimes N) \circ (\varrho_M \otimes \varrho_N). \tag{31}$$

This makes the category of left H-comodules into a monoidal category $({}^{\mathsf{H}}\mathsf{Comod}, \otimes, K)$ but not in a symmetric monoidal category because the product of H it is not associative in general.

In a similar way we can define the categories of right H-quasimodules, right H-modules, and right H-comodules. All of them are monoidal with the corresponding diagonal actions an coactions. The notations for these categories will be $\mathsf{QMod}_{\mathsf{H}}$, $\mathsf{Mod}_{\mathsf{H}}$ and $\mathsf{Mod}^{\mathsf{H}}$ respectively. As in the left case $\mathsf{QMod}_{\mathsf{H}}$, $\mathsf{Mod}_{\mathsf{H}}$ are symmetric monoidal categories, and then with product, when H is cocommutative.

If D is a Hopf coquasigroup we will say that (M, ϱ_M) is a left D-quasicomodule if M is an object in C and $\varrho_M : M \to D \otimes M$ is a morphism in C satisfying (28) for $H = D$ and

$$\begin{aligned}(\mu_D \otimes M) \circ (D \otimes \lambda_D \otimes M) \circ (D \otimes \varrho_M) \circ \varrho_M \\ = \eta_D \otimes M = (\mu_D \otimes M) \circ (\lambda_D \otimes \varrho_M) \circ \varrho_M.\end{aligned} \tag{32}$$

If (M, ϱ_M) satisfies (29) instead of (32) we will say that (M, ϱ_M) is a left D-comodule. With ${}^{\mathsf{D}}\mathsf{QComod}$ we will denote the category of left D-quasicomodules and with ${}^{\mathsf{D}}\mathsf{Comod}$ the category of left D-comodules. Note that (D, δ_D) is an example of D-quasicomodule but not an example of D-comodule. In these categories the morphisms are the ones satisfying (30), for $H = D$, and they are monoidal categories with the product induced by the diagonal coaction (31). They are symmetric monoidal categories, and then with product, if D is commutative. On the other hand, the notion of left D-module is defined as in the case of Hopf quasigroups and then we will denote the category of left D-modules by ${}_{\mathsf{D}}\mathsf{Mod}$. This category with the diagonal product is monoidal but not with product in general. Finally, we can define the corresponding categories of right objects denoted by $\mathsf{QComod}_{\mathsf{D}}$, $\mathsf{Comod}_{\mathsf{D}}$ and $\mathsf{Mod}^{\mathsf{D}}$ respectively.

Definition 6.2. Let D be a symmetric monoidal category with tensor product $\boxtimes$, unit I and symmetry τ. An object M in D is said to be invertible if there exist $\widehat{M}$, an object in D, and an isomorphism in D $\omega_M : M\otimes \widehat{M}\to I$.

The object M is invertible if and only if the functor $M\otimes - : \mathsf{D}\to\mathsf{D}$ is an equivalence of categories. In this setting if we put $F=M\otimes -$ and $G=\widehat{M}\otimes -$ we have that the natural isomorphisms $a_M : Id_{\mathsf{D}} \Longrightarrow GF$ and $b_M : FG\Longrightarrow Id_{\mathsf{D}}$ associated to the equivalence are defined by

$$a_M(N)=(\tau_{M,\widehat{M}}\circ\omega_M^{-1})\otimes N,\;\; b_M(N)=\omega_M\otimes N$$

for all N in D. It is well-known that this equivalence induces an adjoint equivalence by replacing either one of the natural isomorphism a_M, b_M by a new unit or counit. In this case the unit $u_M : Id_{\mathsf{D}}\Longrightarrow GF$ and the counit $v_M : FG\Longrightarrow Id_{\mathsf{D}}$ of the adjunction $F\dashv G$ are defined by

$$u_M(N)=a_M(N),$$

$$v_M(N)=b_M(N)\circ(M\otimes a_M^{-1}(K)\otimes\widehat{M}\otimes N)\circ(M\otimes\widehat{M}\otimes b_M^{-1}(N)),$$

for all N in $\mathcal{D}$. Therefore, M is finite in D and we can assume that $M^*=\widehat{M}$. Moreover, If we define $\Delta_M=n_M^{M^*}\circ v_M^{-1}(K)$, by (4) we obtain that $\nabla_M\circ\Delta_M = v_M(I)\circ v_M(I)^{-1}=id_I$ and therefore we have that M is a progenerator in D.

With $Pic(\mathsf{D})$ we will denote the set of isomorphism classes $[M]$ of invertible objects in D. The set $Pic(\mathsf{D})$ is an abelian group under the operation induced by the tensor product. The unit element is $[I]$ and the inverse of $[M]$ is $[M^*]$. Also, note that if $\mathsf{Pic}(\mathsf{D})$ denotes the full subcategory of D whose objects are the invertible objects in D we have that $\mathsf{Pic}(\mathsf{D})$ is a category with product and $\{I\}$ is a cofinal subcategory of $\mathsf{Pic}(\mathsf{D})$. Therefore $K_1\mathsf{Pic}(\mathsf{D})\cong Aut_{\mathsf{D}}(I)$ and $K_0\mathsf{Pic}(\mathsf{D})\cong Pic(\mathsf{D})$.

Remark 6.3. Let H be a cocommutative Hopf quasigroup in the category C. We know that ${}_{\mathsf{H}}\mathsf{QMod}$ and ${}_{\mathsf{H}}\mathsf{Mod}$ are symmetric monoidal categories and, as a consequence, they are categories with product. With $Pic({}_{\mathsf{H}}\mathsf{QMod})$ and $Pic({}_{\mathsf{H}}\mathsf{Mod})$ we shall denote the Picard groups of ${}_{\mathsf{H}}\mathsf{QMod}$ and ${}_{\mathsf{H}}\mathsf{Mod}$ respectively. Then, if $\mathsf{Pic}({}_{\mathsf{H}}\mathsf{QMod})$ and $\mathsf{Pic}({}_{\mathsf{H}}\mathsf{Mod})$ are the categories with product of the invertible objects in ${}_{\mathsf{H}}\mathsf{QMod}$ and ${}_{\mathsf{H}}\mathsf{Mod}$, we have that $\{(K,\varphi_K=\varepsilon_H\otimes K)\}$ is a cofinal subcategory of $\mathsf{Pic}({}_{\mathsf{H}}\mathsf{QMod})$ and $\mathsf{Pic}({}_{\mathsf{H}}\mathsf{Mod})$. Therefore,

$$K_1\mathsf{Pic}({}_{\mathsf{H}}\mathsf{QMod})\cong Aut_{\mathsf{C}}(K)\cong K_1\mathsf{Pic}({}_{\mathsf{H}}\mathsf{Mod}),$$

$$K_0\mathsf{Pic}({}_{\mathsf{H}}\mathsf{QMod})\cong Pic({}_{\mathsf{H}}\mathsf{QMod}),\quad K_0\mathsf{Pic}({}_{\mathsf{H}}\mathsf{Mod})\cong Pic({}_{\mathsf{H}}\mathsf{Mod}),$$

and there exists a group injection $i_H : Pic({}_{\mathsf{H}}\mathsf{Mod})\to Pic({}_{\mathsf{H}}\mathsf{QMod})$.

Proposition 6.4. *Let H be a finite cocommutative Hopf quasigroup in $\mathcal{C}$.*

(i) *If* (M, φ_M) *is a left* H*-quasimodule,*

$$(M, \rho_M = c_{H^*,M} \circ (H^* \otimes \varphi_M) \circ (\alpha_H(K) \otimes M))$$

is a right H^**-quasicomodule.*

(ii) *If* (M, φ_M) *is a left* H*-module,*

$$(M, \rho_M = c_{H^*,M} \circ (H^* \otimes \varphi_M) \circ (\alpha_H(K) \otimes M))$$

is a right H^**-comodule.*

Proof. (i) Note that by the naturality of c, the triangular equalities for the unit α_H and counit β_H of the adjunction $H \otimes - \dashv H^* \otimes -$, and (24) we have

$$(M \otimes \varepsilon_{H^*}) \circ \rho_M = (\beta_H(K) \otimes \varphi_M) \circ (\eta_H \otimes \alpha_H(K) \otimes M) = \varphi_M \circ (\eta_H \otimes M) = id_M.$$

On the other hand,

$$\begin{aligned}
&(M \otimes \mu_{H^*}) \circ (\rho_M \otimes \lambda_{H^*}) \circ \rho_M \\
&= (M \otimes ((H^* \otimes \beta_H(K)) \circ (H^* \otimes ((\lambda_H \otimes \beta_H(K)) \circ (\delta_H \otimes H^*)) \otimes H^*) \\
&\quad \circ (\alpha_H(K) \otimes H^* \otimes H^*))) \circ (c_{H^*,M} \otimes H^*) \circ (H^* \otimes \varphi_M \otimes H^*) \\
&\quad \circ (\alpha_H(K) \otimes c_{H^*,M}) \circ (H^* \otimes \varphi_M) \circ (\alpha_H(K) \otimes M) \text{ (triangular equalities)} \\
&= ((\varphi_M \circ (H \otimes \varphi_M)) \otimes H^*) \circ (H \otimes H \otimes c_{H^*,M}) \circ (H \otimes c_{H^*,H} \otimes M) \\
&\quad \circ (c_{H^*,H} \otimes \beta_H(K) \otimes H \otimes M) \circ (H^* \otimes c_{H,H} \otimes \alpha_H(K) \otimes M) \\
&\quad \circ (H^* \otimes \lambda_H \otimes \beta_H(K) \otimes H \otimes M) \circ (H^* \otimes \delta_H \otimes \alpha_H(K) \otimes M) \\
&\quad \circ (\alpha_H(K) \otimes M) \text{ (naturality of } c) \\
&= ((\varphi_M \circ (H \otimes \varphi_M)) \otimes H^*) \circ (H \otimes H \otimes c_{H^*,M}) \circ (H \otimes c_{H^*,H} \otimes M) \\
&\quad \circ (c_{H^*,H} \otimes H \otimes M) \circ (H^* \otimes ((H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H) \otimes M) \\
&\quad \circ (\alpha_H(K) \otimes M) \text{ (triangular equalities)} \\
&= c_{H^*,M} \circ (H^* \otimes ((\varphi_M \circ (H \otimes \varphi_M) \circ (H \otimes \lambda_H \otimes M) \circ (\delta_H \otimes M))) \\
&\quad \circ (\alpha_H(K) \otimes M) \text{ (naturality of } c \text{ and cocommutativity of } H) \\
&= M \otimes \eta_{H^*} \text{ ((25) and naturality of } c).
\end{aligned}$$

The proof for $(M \otimes \mu_{H^*}) \circ (M \otimes \lambda_{H^*} \otimes H^*) \circ (\rho_M \otimes H^*) \otimes \rho_M = M \otimes \eta_{H^*}$ uses the same pattern and we leave the details to the reader. Finally (ii) follows by (27) in a similar way. □

The following result generalizes the one proved by Caenepeel in [9] (see also [10, Proposition 13.3.1]) for left modules associated to a faithfully projective commutative and cocommutative Hopf algebra.

Theorem 6.5. *Let* H *be a finite cocommutative Hopf quasigroup in* $\mathcal{C}$*. Then*

$$Pic({}_{\mathsf{H}}\mathsf{Mod}) \cong Pic(\mathsf{C}) \oplus G(H^*).$$

Proof. In the proof of this result the commutativity and the associativity of the product on H is not needed. As a consequence, the proof is similar to the one developed in [12, Proposition 3.11]. For the convenience of the reader we give a brief resume. First, by the previous proposition we have that if (M, φ_M) is a left H-module, then $(M, \rho_M = c_{H^*,M} \circ (H^* \otimes \varphi_M) \circ (\alpha_H(K) \otimes M))$ is a right H^*-comodule. Let (M, φ_M) be a left H-module invertible. The morphism $h_M = (u_M^{-1}(K) \otimes H^*) \circ (M^* \otimes \rho_M) \circ u_M(K)$ is in $G(H^*)$. As a consequence, it is possible to define the map

$$w : Pic({}_{\mathsf{H}}\mathsf{Mod}) \to G(H^*)$$

by $w([(M, \varphi_M)]) = h_M$. This map is a group morphism and it is an epimorphism because if $h : K \to H^*$ is in $G(H^*)$, $h = w([(K, \varphi_K = \beta_H(K) \circ (H \otimes h))])$. Note that the object $(K, \varphi_K = \beta_H(K) \circ (H \otimes h))$ is invertible in ${}_{\mathsf{H}}\mathsf{Mod}$ with $(\widehat{K} = K, \varphi_{\widehat{K}} = \beta_H(K) \circ (\lambda_H \otimes h))$.

On the other hand, there exists a morphism of groups

$$pic : Pic(\mathsf{C}) \to Pic({}_{\mathsf{H}}\mathsf{Mod})$$

defined by $pic([M]) = [(M, \varepsilon_H \otimes M)]$. If $w([(M, \varphi_M)]) = \eta_{H^*}$ it is easy to prove that $\varphi_M = \varepsilon_H \otimes M$ and then $[(M, \varphi_M)] \in Im(pic)$. Moreover, $w(pic([M])) = \eta_{H^*}$ for all M invertible in C. Therefore, the sequence

$$0 \to Pic(\mathsf{C}) \to Pic({}_{\mathsf{H}}\mathsf{Mod}) \to G(H^*) \to 0$$

is exact and splits because there exists a morphism

$$p_H : Pic({}_{\mathsf{H}}\mathsf{Mod}) \to Pic(\mathsf{C})$$

defined by

$$p_H([(M, \varphi_M)]) = [M]$$

such that $p_H \circ pic = id_{Pic(\mathsf{C})}$. □

Proposition 6.6. *Let* C *and* D *be symmetric monoidal categories, let* $F : \mathsf{C} \to \mathsf{D}$ *be a strong symmetric monoidal functor, and let* H *be a Hopf quasigroup in* C. *The following assertions hold.*

(i) *If* (M, φ_M) *is a left* H*-quasimodule,* $(F(M), \varphi_{F(M)} = F(\varphi_M) \circ \Phi_{H,M})$ *is a left* $F(H)$*-quasimodule.*
(ii) *If* (M, φ_M) *is a left* H*-module,* $(F(M), \varphi_{F(M)} = F(\varphi_M) \circ \Phi_{H,M})$ *is a left* $F(H)$*-module.*
(iii) *If* $f : M \to N$ *is a morphism of left* H*-(quasi)modules in* C*,* $F(f)$ *is a morphism of left* $F(H)$*-(quasi)modules in* D*.*
(iv) *If* (M, φ_M)*,* (N, φ_N) *are left* H*-(quasi)modules,* $\Phi_{M,N}$ *is a morphism of left* $F(H)$*-(quasi)modules.*
(v) *The morphism* Φ_0 *is a morphism of left* $F(H)$*-(quasi)modules for*

$$\varphi_{F(K)} = F(\varepsilon_H \otimes K) \circ \Phi_{H,K}$$

and $\varphi_I = \varepsilon_{F(H)} \otimes I$*.*

Proof. As in the previous cases we shall assume that $\boxtimes$ is the tensor product of D and I its unit object. The proof for (i) is the following: Firstly note that, by the naturality of Φ, (24), and (11), we have:

$$\varphi_{F(M)} \circ (\eta_{F(H)} \boxtimes F(M)) = \Phi_{K,M} \circ (\Phi_0 \boxtimes F(M)) = id_{F(M)}.$$

On the other hand,

$$\begin{aligned}
&\varphi_{F(M)} \circ (\lambda_{F(H)} \boxtimes \varphi_{F(M)}) \circ (\delta_{F(H)} \boxtimes F(M)) \\
&= F(\varphi_M \circ (\lambda_H \otimes \varphi_M)) \circ \Phi_{H,H\otimes M} \circ (F(H) \boxtimes \Phi_{H,M}) \\
&\quad \circ ((\Phi^{-1}_{H,H} \circ F(\delta_H)) \boxtimes F(M)) \text{ (naturality of } \Phi) \\
&= F(\varphi_M \circ (\lambda_H \otimes \varphi_M)) \circ \Phi_{H,H\otimes M} \circ \Phi^{-1}_{H,H\otimes M} \circ \Phi_{H\otimes H,M} \\
&\quad \circ (F(\delta_H) \boxtimes F(M)) \text{ ((9))} \\
&= F(\varphi_M \circ (\lambda_H \otimes \varphi_M) \circ (\delta_M \otimes M)) \circ \Phi_{H,M} \text{ (naturality of } \Phi) \\
&= F(\varepsilon_H \otimes M) \circ \Phi_{H,M} \text{ ((25))} \\
&= \Phi_{K,M} \circ (F(\varepsilon_H) \boxtimes F(M)) \text{ (naturality of } \Phi) \\
&= \varepsilon_{F(H)} \boxtimes F(M) \text{ ((11))}.
\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
&\varphi_{F(M)} \circ (F(H) \boxtimes \varphi_{F(M)}) \circ (F(H) \boxtimes \lambda_{F(H} \boxtimes F(M)) \circ (\delta_{F(H)} \boxtimes F(M)) \\
&= \varepsilon_{F(H)} \boxtimes F(M).
\end{aligned}$$

The proof of (ii) follows by analogous arguments and we leave the details to the reader.

If $f : M \to N$ is a morphism of left H-(quasi)modules, by the naturality of Φ, we obtain that $F(f)$ is a morphism of left $F(H)$-(quasi)modules in D. Then (iii) holds. Similarly, by the naturality of Φ and (i) of Definition 2.5, it is easy to prove (iv). Finally (v) follows from (11). $\square$

Corollary 6.7. *In the conditions of the previous proposition, if* (M, φ_M) *is an invertible left* H*-(quasi)module,* $(F(M), \varphi_{F(M)})$ *is an invertible left* $F(H)$*-(quasi)module.*

Proof. If (M, φ_M) is an invertible left H-(quasi)module and $(\widehat{M}, \varphi_{\widehat{M}})$ is the left H-(quasi)module such that there exists an isomorphism $\omega_M : M \otimes \widehat{M} \to K$, we can choose the left $F(H)$-(quasi)module $(\widehat{F(M)} = F(\widehat{M}), \varphi_{\widehat{F(M)}} = \varphi_{F(\widehat{M})})$ and then, by (iii)-(v) of the previous proposition we have that

$$\omega_{F(M)} = \Phi_0^{-1} \circ F(\omega_M) \circ \Phi_{M,\widehat{M}} : F(M) \boxtimes \widehat{F(M)} \to I$$

is an isomorphism of left $F(H)$-(quasi)modules. Therefore $(F(M), \varphi_{F(M)})$ is an invertible left $F(H)$-(quasi)module and the assertion holds. $\square$

Corollary 6.8. *In the conditions of Proposition* 6.6*, if H is cocommutative, there exists cofinal product preserving functors*

$$QPic(F) : \mathsf{Pic}({}_{\mathsf{H}}\mathsf{QMod}) \to \mathsf{Pic}({}_{\mathsf{F(H)}}\mathsf{QMod}),$$

$$Pic(F) : \mathsf{Pic}({}_{\mathsf{H}}\mathsf{Mod}) \to \mathsf{Pic}({}_{\mathsf{F(H)}}\mathsf{Mod})$$

defined as $(M, \varphi_M) \rightsquigarrow (F(M), \varphi_{F(M)})$ on objects and in the obvious way on morphisms.

Proof. The proof follows directly from Corollary 6.7. Note that if (P, φ_P) is an invertible left $F(H)$-(quasi)module, then, by (v) of Proposition 6.6, $P \boxtimes \widehat{P}$ is isomorphic to $F(K)$ as left $F(H)$-(quasi)modules.

□

Theorem 6.9. *In the conditions of Proposition* 6.6*, if H is cocommutative, there exist two exact sequences*

$$Aut_{\mathsf{C}}(K) \to Aut_D(I) \to K_1\Phi(QPic(F)) \to Pic({}_{\mathsf{H}}\mathsf{QMod}) \to Pic({}_{\mathsf{F(H)}}\mathsf{QMod}),$$

$$Aut_{\mathsf{C}}(K) \to Aut_D(I) \to K_1\Phi(Pic(F)) \to Pic({}_{\mathsf{H}}\mathsf{Mod}) \to Pic({}_{\mathsf{F(H)}}\mathsf{Mod}).$$

Proof. The proof is a consequence of the general results about exact sequences associated to cofinal product preserving functors of Corollary 6.8 and the isomorphisms of Remark 6.3. □

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José N. Alonso Álvarez
Departamento de Matemáticas
Universidad de Vigo
Campus Universitario Lagoas-Marcosende
E-36280 Vigo, Spain
Email address: jnalonso@uvigo.es

José M. Fernández Vilaboa
Departamento de Matemáticas
Universidad de Santiago de Compostela
E-15771 Santiago de Compostela, Spain
Email address: josemanuel.fernandez@usc.es

Ramón González Rodríguez
Departamento de Matemática Aplicada II
Universidad de Vigo
Campus Universitario Lagoas-Marcosende
E-36310 Vigo, Spain
Email address: rgon@dma.uvigo.es