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SOME FINITENESS RESULTS FOR CO-ASSOCIATED PRIMES OF GENERALIZED LOCAL HOMOLOGY MODULES AND APPLICATIONS

Yen Ngoc Do, Tri Minh Nguyen, and Nam Tuan Tran

Abstract. We prove some results about the finiteness of co-associated primes of generalized local homology modules inspired by a conjecture of Grothendieck and a question of Huneke. We also show some equivalent properties of minimax local homology modules. By duality, we get some properties of Herzog's generalized local cohomology modules.

1. Introduction

In [13] J. Herzog introduced the definition of generalized local cohomology which is an extension of local cohomology of A. Grothendieck ([4, 11]). Let I be an ideal of a noetherian commutative ring R and M, N R-modules. In [20] we defined the i-th generalized local homology module $H_i^I(M,N)$ of M, N with respect to I by

$$H_i^I(M,N) = \varprojlim_t \operatorname{Tor}_i^R(M/I^tM, N).$$

This definition is in some sense dual to J. Herzog's definition of generalized local cohomology modules [13] and in fact a generalization of the usual local homology modules ([5, 6]).

In [10] A. Grothendieck gave a conjecture: For any ideal I of R and any finitely generated R-module M, the module $\operatorname{Hom}_R(R/I, H_I^i(M))$ is finitely generated for all i. A weaker question is due to C. Huneke [14]: If M is finitely generated, is the number of associated primes of local cohomology modules $H_I^i(M)$ always finite? Hartshorne in [12] gives a counterexample to Grothendieck's conjecture and Singh in [25] gives a counterexample to C. Huneke's question. In [16] M. Katzman also gives an example of an infinite set of associated primes

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of a local cohomology module. Brodmann-Faghani [3] and Aghapournahr-Melkersson [1] also gave some finiteness results for associated primes of local cohomology modules.

There is a similar question in (generalized) local homology theory: When is the set of co-associated primes of (generalized) local homology modules finite? Note that the finiteness of co-associated primes and associated primes of (generalized) local homology modules and (generalized) local cohomology modules is closely related to the local-global-principle for finiteness dimensions of G. Faltings [9]. T. T. Nam in [23] showed some finiteness results for co-associated primes of local homology modules. In this paper we extend some results of [23] to generalized local homology modules. The purpose of this paper is to study the finiteness of co-associated primes of generalized local homology modules inspired by Grothendieck's conjecture and Huneke's question. We also study some properties of minimax local homology modules. By duality, we get some properties of Herzog's generalized local cohomology modules. The organization of the paper is as follows.

In Section 2 we recall briefly some properties of linearly compact modules and (generalized) local homology modules that we shall use.

In Section 3 we establish some properties of minimax local homology modules. Theorems 3.3, 3.4 and 3.5 give us some results on the finiteness of co-associated primes of local homology modules inspired by A. Grothendieck's conjecture and C. Huneke's question. In [20] and [24] we also showed some similar results, but the new results in this paper are stronger. In addition we show some equivalent properties of minimax local homology modules (Theorem 3.6).

In the last section, based on the dual theorem (Theorem 4.5) and the results of generalized local homology modules in the previous section we get the finiteness results of associated primes of generalized local cohomology modules (Theorems 4.7, 4.9 and 4.12). The section is closed by Theorem 4.14 in which we show some equivalent properties of minimax generalized local cohomology modules.

Throughout this paper, R is a noetherian commutative ring has a topological structure.

2. Preliminaries

We begin by recalling the concept of *linearly compact* defined by I. G. Macdonald [17]. A Hausdorff linearly topologized R-module M is said to be *linearly compact* if for any family $\mathcal{F}$ of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection. A Hausdorff linearly topologized R-module M is called *semi-discrete* if every submodule of M is closed. Thus a discrete R-module is semi-discrete. It is clear that artinian R-modules are linearly compact with the discrete topology. So the class of semi-discrete linearly compact modules contains all artinian modules. Moreover, if $(R, \mathfrak{m})$ is a complete ring, then the finitely generated R-modules are also linearly compact and discrete. Following are some basic properties of linearly compact modules.

Lemma 2.1 ([17, 3.5, 3.14, 3.15])**.** (i) *Let M be a Hausdorff linearly topologized R-module, N a closed submodule of M. Then M is linearly compact if and only if N and M/N are linearly compact.*

(ii) *If M is a linearly compact module, then for each positive integer t, I^tM is a closed submodule of M. Moreover, $I(\bigcap_{t>0} I^tM) = \bigcap_{t>0} I^tM$.*

Denoting by $\varprojlim_t^i$ the i-th right derived functor of the inverse limit $\varprojlim_t$, we have the following lemma.

Lemma 2.2 (see [15, 7.1])**.** *Let $\{M_t\}$ be an inverse system of linearly compact modules with continuous homomorphisms. Then $\varprojlim_t^i M_t = 0$ for all $i > 0$. Therefore, if*

$$0 \longrightarrow \{M_t\} \longrightarrow \{N_t\} \longrightarrow \{P_t\} \longrightarrow 0$$

is a short exact sequence of inverse systems of R-modules provided $\{M_t\}$ is an inverse system of linearly compact modules with continuous homomorphisms, then the sequence of inverse limits

$$0 \longrightarrow \varprojlim_t M_t \longrightarrow \varprojlim_t N_t \longrightarrow \varprojlim_t P_t \longrightarrow 0$$

is exact.

Lemma 2.3 ([6, 2.7])**.** *If M is a finitely generated R-module and $\{N_s\}$ is an inverse system of linearly compact R-modules with continuous homomorphisms, then for all $i \geq 0$, $\{\mathrm{Tor}_i^R(M, N_s)\}$ forms an inverse system of linearly compact modules with continuous homomorphisms. Moreover, we have*

$$\mathrm{Tor}_i^R(M, \varprojlim_s N_s) \cong \varprojlim_s \mathrm{Tor}_i^R(M, N_s).$$

Let I be an ideal of R, the I-adic completion $\Lambda_I(M)$ of an R-module M is defined by $\Lambda_I(M) = \varprojlim_t M/I^tM$. In [5], the i-th local homology module $H_i^I(M)$ of an R-module M with respect to I is defined by

$$H_i^I(M) \cong \varprojlim_t \mathrm{Tor}_i^R(R/I^t, M).$$

It is clear that $H_0^I(M) \cong \Lambda_I(M)$.

Lemma 2.4 ([6, 3.3, 3.10, 4.1])**.** *Let M be a linearly compact R-module. Then*

(i) *$H_i^I(M)$ is also a linearly compact R-module for all $i \geq 0$.*

(ii) $H_i^I(\bigcap_{t>0} I^tM) \cong \begin{cases} 0, & i = 0, \\ H_i^I(M), & i > 0. \end{cases}$

(iii) *Assume in addition that M is a semi-discrete linearly compact R-module. Then $H_0^I(M) = 0$ if and only if $xM = M$ for some $x \in I$.*

Let M and N be R-modules. In [20], the i-th generalized local homology module $H_i^I(M, N)$ of M, N with respect to I is defined by

$$H_i^I(M, N) = \varprojlim_t \operatorname{Tor}_i^R(M/I^t M, N).$$

When $i = 0$, $H_0^I(M, N) \cong \Lambda_I(M, N)$ in which

$$\Lambda_I(M, N) = \varprojlim_t (R/I^t \otimes_R M \otimes_R N).$$

In particular, $H_i^I(R, N) = H_i^I(N)$.

Lemma 2.5 ([28, 2.3(i)]). *If M is a finitely generated R-module and N is a linearly compact R-module, then for all $i \geq 0$, $H_i^I(M, N)$ is a linearly compact R-module.*

Lemma 2.6 ([28, 3.4]). *Let M be a finitely generated module and N a linearly compact R-module. If N is complete in the I-adic topology (i.e., $\Lambda_I(N) \cong N$), then for all $i \geq 0$, there is an isomorphism*

$$\operatorname{Tor}_i^R(M, N) \cong H_i^I(M, N).$$

The co-support $\operatorname{Cosupp}_R(M)$ of an R-module M is the set of primes $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\operatorname{Ann}(L) \subseteq \mathfrak{p}$ ([27, 2.1]). Note that a module is cocyclic if it is a submodule of $E(R/\mathfrak{m})$.

If $0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$ is an exact sequence of R-modules, then

$$\operatorname*{Cosupp}_R(M) = \operatorname*{Cosupp}_R(N) \cup \operatorname*{Cosupp}_R(K) \text{ ([27, 2.7]).}$$

Lemma 2.7 ([21, 3.14] and [18, 3.8]). *Let M be a finitely generated R-module and N a linearly compact R-module. Then*

$$\operatorname{Cosupp}_R(H_i^I(M, N)) \subseteq \operatorname{Supp}(M) \bigcap \operatorname{Cosupp}_R(N) \bigcap V(I)$$

for all $i \geq 0$.

A prime ideal $\mathfrak{p}$ is called *co-associated* to a non-zero R-module M if there is an artinian homomorphic image T of M with $\mathfrak{p} = \operatorname{Ann}_R T$ [27]. The set of co-associated primes of M is denoted by $\operatorname{Coass}_R(M)$. It follows that $\operatorname{Coass}_R(M) \subseteq \operatorname{Cosupp}_R(M)$. If $0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$ is an exact sequence of R-modules, then $\operatorname{Coass}_R(K) \subseteq \operatorname{Coass}_R(M) \subseteq \operatorname{Coass}_R(N) \cup \operatorname{Coass}_R(K)$. If M is a semi-discrete linearly compact R-module, then the set $\operatorname{Coass}(M)$ is finite ([6, 2.9]).

Lemma 2.8 ([27, 1.21]). *Let M be a finitely generated R-module and N an R-module. Then*

$$\operatorname{Coass}_R(M \otimes_R N) = \operatorname{Supp}_R(M) \bigcap \operatorname{Coass}_R(N).$$

3. The co-associated primes of generalized local homology modules

We first recall the concept of *minimax module* introduced by H. Zöschinger [30]. An R-module M is called a *minimax module* if there is a finitely generated submodule N of M such that the quotient module M/N is artinian. Thus the class of minimax modules includes all finitely generated and all artinian modules. Moreover, it also includes all semi-discrete linearly compact modules.

An R-module M is said to be I-coartinian if $\mathrm{Cosupp}_R(M) \subseteq V(I)$ and $\mathrm{Tor}_i^R(R/I, M)$ is an artinian R-module for each i [18]. This definition is in some sense dual to R. Hartshone's concept of I-cofinite modules [12]. In order to state Theorem 3.3 the following lemma will be useful.

Lemma 3.1 ([23, 3.5])**.** *Let M be a minimax linearly compact R-module with* $\mathrm{Cosupp}(M) \subseteq V(I)$*. Then the R-module M is I-coartinian if and only if M/IM is artinian.*

Lemma 3.2 ([23, 3.6])**.** *Let M be an I-coartinian minimax linearly compact R-module. If N is a closed submodule of M, then N and M/N are I-coartinian minimax linearly compact R-modules.*

The following theorem is inspired by A. Grothendieck's conjecture [10] and C. Huneke's question [14].

Theorem 3.3. *Let M be a finitely generated R-module and N a semi-discrete linearly compact R-module such that $N/(\bigcap_{t>0} I^t N)$ is an artinian R-module. Let s be a non-negative integer. If $H_i^I(M,N)$ is minimax for all $i < s$, then $H_i^I(M,N)$ is I-coartinian for all $i < s$ and $H_s^I(M,N)/IH_s^I(M,N)$ is artinian. In particular,* $\mathrm{Coass}(H_s^I(M,N))$ *is a finite set.*

Proof. The proof is by induction on s. When $s = 0$, it is trivial that $H_i^I(M,N)$ is I-coartinian for all $i < 0$. By Lemma 2.2 the short exact sequence of inverse systems of linearly compact modules

$$0 \longrightarrow \{I^t N\} \longrightarrow \{N\} \longrightarrow \{N/I^t N\} \longrightarrow 0$$

induces a short exact sequence

$$0 \longrightarrow \bigcap_{t>0} I^t N \longrightarrow N \longrightarrow \Lambda_I(N) \longrightarrow 0.$$

Set $K = \bigcap_{t>0} I^t N$, then $\Lambda_I(N) \cong N/K$ which is an artinian R-module. From Lemma 2.3 we have

$$\begin{aligned}\Lambda_I(M,N) &= \varprojlim_t (R/I^t \otimes_R M \bigotimes_R N) \\ &\cong \varprojlim_t (M \otimes_R N/I^t N) \cong M \otimes_R \Lambda_I(N).\end{aligned}$$

It follows that $\Lambda_I(M,N)$ is artinian and then $\Lambda_I(M,N)/I\Lambda_I(M,N)$ is also artinian.

Let $s > 0$. It follows from the inductive hypothesis that $H_i^I(M,N)$ is I-coartinian for all $i < s-1$ and $H_{s-1}^I(M,N)/IH_{s-1}^I(M,N)$ is artinian. Moreover, $\operatorname{Cosupp}(H_{s-1}^I(M,N)) \subseteq V(I)$ by Lemma 2.7. We conclude from Lemma 3.1 that $H_{s-1}^I(M,N)$ is I-coartinian. Now the short exact sequence of linearly compact R-modules

$$0 \longrightarrow K \longrightarrow N \longrightarrow N/K \longrightarrow 0$$

gives rise to a long exact sequence of generalized local homology modules

$$(*) \qquad \cdots \longrightarrow H_s^I(M,K) \longrightarrow H_s^I(M,N) \xrightarrow{\alpha} H_s^I(M,N/K) \longrightarrow \cdots .$$

It induces an exact sequence

$$H_s^I(M,K) \longrightarrow H_s^I(M,N) \longrightarrow \operatorname{Im}\alpha \longrightarrow 0.$$

Then we have the following exact sequence

$$H_s^I(M,K)/IH_s^I(M,K) \longrightarrow H_s^I(M,N)/IH_s^I(M,N) \longrightarrow \operatorname{Im}\alpha/I\operatorname{Im}\alpha \longrightarrow 0.$$

Since N/K is complete in I-adic topology, there is an isomorphism

$$\operatorname{Tor}_i^R(M,N/K) \cong H_i^I(M,N/K)$$

for all $i \geq 0$ by Lemma 2.6. So $H_s^I(M,N/K)$ is an artinian R-module and then $\operatorname{Im}\alpha/I\operatorname{Im}\alpha$ is also an artinian R-module. In order to prove the artinianness of $H_s^I(M,N)/IH_s^I(M,N)$ we only need to prove that $H_s^I(M,K)/IH_s^I(M,K)$ is an artinian R-module.

By Lemma 2.4(ii), (iii) there is an element $x \in I$ such that $xK = K$. Now the short exact sequence

$$0 \longrightarrow 0 :_K x \longrightarrow K \xrightarrow{.x} K \longrightarrow 0$$

gives rise to a long exact sequence

$$\cdots \longrightarrow H_i^I(M,K) \xrightarrow{.x} H_i^I(M,K) \longrightarrow H_{i-1}^I(M, 0 :_K x) \longrightarrow \cdots .$$

Thus we have an induced short exact sequence

$$0 \longrightarrow H_i^I(M,K)/xH_i^I(M,K) \longrightarrow H_{i-1}^I(M,0 :_K x) \longrightarrow 0 :_{H_{i-1}^I(M,K)} x \longrightarrow 0.$$

According to the above argument, $H_i^I(M,N/K)$ is an artinian R-module for all ≥ 0, so $H_i^I(M,N/K)$ is minimax. As $H_i^I(M,N)$ is minimax for all $i < s$, it follows from the exact sequence $(*)$ that $H_i^I(M,K)$ is minimax for all $i < s$. Then $H_i^I(M,K)/xH_i^I(M,K)$ and $0 :_{H_{i-1}^I(M,K)} x$ are minimax for all $i < s$. Hence $H_{i-1}^I(M, 0 :_K x)$ is also minimax for all $i < s$. It should be noted by [29, 1 (b0)] that $0 :_K x$ is an artinian R-module. From the inductive hypothesis, $H_{i-1}^I(M, 0 :_K x)$ is I-coartinian for all $i < s$ and $H_{s-1}^I(M, 0 :_K x)/IH_{s-1}^I(M, 0 :_K x)$ is artinian. We now have $\operatorname{Cosupp}(H_{s-1}^I(M, 0 :_K x)) \subseteq V(I)$ by Lemma 2.7 and $H_{s-1}^I(M, 0 :_K x)$ is a minimax linearly compact module. Therefore, $H_{s-1}^I(M, 0 :_K x)$ is I-coartinian by Lemma 3.1 and then

$0 :_{H^I_{s-1}(M,K)} x$ is I-coartinian by Lemma 3.2. Now, the last exact sequence induces an exact sequence

$$\mathrm{Tor}_1^R(R/I, 0 :_{H^I_{s-1}(M,K)} x) \to H^I_s(M,K)/IH^I_s(M,K) \to H^I_{s-1}(M, 0 :_K x)/IH^I_{s-1}(M, 0 :_K x).$$

As $\mathrm{Tor}_1^R(R/I, 0 :_{H^I_{s-1}(M,K)} x)$ is artinian, $H^I_s(M,K)/IH^I_s(M,K)$ is also artinian. Thus $H^I_s(M,N)/IH^I_s(M,N)$ is artinian and then

$$\mathrm{Coass}(H^I_s(M,N)/IH^I_s(M,N))$$

is a finite set. We also have by Lemma 2.7

$$\mathrm{Coass}(H^I_s(M,N)) \subseteq \mathrm{Cosupp}(H^I_s(M,N)) \subseteq V(I).$$

Therefore

$$\begin{aligned}\mathrm{Coass}(H^I_s(M,N)/IH^I_s(M,N)) &= \mathrm{Coass}(H^I_s(M,N)) \cap V(I)\\ &= \mathrm{Coass}(H^I_s(M,N))\end{aligned}$$

by Lemma 2.8. The proof is complete. □

The following theorem gives us a more general result.

Theorem 3.4. *Let M be a finitely generated R-module and N a semi-discrete linearly compact R-module such that $N/(\bigcap_{t>0} I^tN)$ is an artinian R-module. Let s be a non-negative integer. If $H^I_i(M,N)$ is minimax for all $i < s$ and G is a closed submodule of $H^I_s(M,N)$ such that $H^I_s(M,N)/G$ is minimax, then G/IG is artinian. In particular,* $\mathrm{Coass}(G)$ *is a finite set.*

Proof. It should be noted by Lemma 2.5 that $H^I_s(M,N)$ is linearly compact. Then $H^I_s(M,N)/G$ is also linearly compact by Lemma 2.1(i). Set $L = H^I_s(M,N)/G$, the short exact sequence

$$0 \longrightarrow G \longrightarrow H^I_s(M,N) \longrightarrow L \longrightarrow 0$$

induces an exact sequence

$$\mathrm{Tor}_1^R(R/I, L) \to G/IG \to H^I_s(M,N)/IH^I_s(M,N) \to L/IL \longrightarrow 0.$$

It follows from Theorem 3.3 that $H^I_s(M,N)/IH^I_s(M,N)$ is artinian, so is L/IL. On the other hand, Lemma 2.7 gives $\mathrm{Cosupp}(L) \subseteq \mathrm{Cosupp}(H^I_s(M,N)) \subseteq V(I)$. Then L is I-coartinian by Lemma 3.1. So $\mathrm{Tor}_1^R(R/I, L)$ is artinian and then G/IG is artinian. Finally, $\mathrm{Coass}(G)$ is a finite set. □

We say that an R-module M satisfies the finite condition for co-associated primes if the set of co-associated primes of any submodule of M is finite. Note that if an R-module M satisfies the finite condition for co-associated primes, then any subquotient of M satisfies the finite condition for co-associated primes ([19, Remark 1]).

Let M be a finitely generated module and N a linearly compact R-module, in [24, 1] we showed that if N and $H^I_i(N)$ satisfy the finite condition for co-associated primes for all $i < s$, then $\mathrm{Coass}(H^I_s(M,N))$ is a finite set. In the following theorem, we give a stronger result in which we show that the module $H_s^I(M,N)/IH_s^I(M,N)$ also satisfies the finite condition for co-associated primes.

Theorem 3.5. *Let M be a finitely generated module and N a semi-discrete linearly compact R-module. Let s be a non-negative integer. If $H_i^I(M,N)$ satisfies the finite condition for co-associated primes for all $i < s$, then the module $H_s^I(M,N)/IH_s^I(M,N)$ also satisfies the finite condition for co-associated primes. In particular,* $\operatorname{Coass}(H_s^I(M,N))$ *is a finite set.*

Proof. We proceed by induction on s. When $s = 0$, the short exact sequence of inverse systems of linearly compact modules

$$0 \longrightarrow \{I^t(M \otimes_R N)\}_t \longrightarrow \{M \otimes_R N\}_t \longrightarrow \{(M \otimes_R N)/I^t(M \otimes_R N)\}_t \longrightarrow 0$$

induces by Lemma 2.2 a short exact sequence

$$0 \longrightarrow \bigcap_{t>0} I^t(M \otimes_R N) \longrightarrow M \otimes_R N \longrightarrow \Lambda_I(M,N) \longrightarrow 0.$$

Note that $(\bigcap_{t>0} I^t(M \otimes_R N))/I(\bigcap_{t>0} I^t(M \otimes_R N)) = 0$ by Lemma 2.1(ii). Then

$$\Lambda_I(M,N)/I\Lambda_I(M,N) \cong (M \otimes_R N)/I(M \otimes_R N).$$

By [19, Remark 1(ii)], $\Lambda_I(M,N)/I\Lambda_I(M,N)$ satisfies the finite condition for co-associated primes.

Let $s > 0$. The short exact sequences of linearly compact modules

$$0 \longrightarrow \bigcap_t I^t N \longrightarrow N \longrightarrow \Lambda_I(N) \longrightarrow 0$$

gives rise to an exact sequence

$$H_s^I(M, \bigcap_t I^t N) \xrightarrow{\varphi} H_s^I(M,N) \xrightarrow{\psi} H_s^I(M, \Lambda_I(N)).$$

Then we have the following exact sequences

$$H_s^I(M, \bigcap_t I^t N) \xrightarrow{\varphi} H_s^I(M,N) \longrightarrow \operatorname{Im}\psi \longrightarrow 0,$$

$$H_s^I(M, \bigcap_t I^t N)/IH_s^I(M, \bigcap_t I^t N) \to H_s^I(M,N)/IH_s^I(M,N) \to \operatorname{Im}\psi/I\operatorname{Im}\psi \to 0.$$

Since $\Lambda_I(N)$ is complete in I-adic topology, there is an isomorphism

$$\operatorname{Tor}_s^R(M, \Lambda_I(N)) \cong H_s^I(M, \Lambda_I(N))$$

by Lemma 2.6. As N satisfies the finite condition for co-associated primes, $\Lambda_I(N)$ also satisfies the finite condition for co-associated primes. By [19, Remark 1(ii)], $\operatorname{Tor}_s^R(M, \Lambda_I(N))$ satisfies the finite condition for co-associated primes. It means that $H^I_s(M, \Lambda_I(N))$ satisfies the finite condition for co-associated primes and then $\operatorname{Im}\psi/I\operatorname{Im}\psi$ satisfies the finite condition for co-associated primes. Thus, it is sufficient to show that

$$H^I_s(M, \bigcap_t I^t N)/IH^I_s(M, \bigcap_t I^t N)$$

satisfies the finite condition for co-associated primes.

Set $K = \bigcap_t I^t N$, by Lemma 2.4(ii), (iii) there exists an element $x \in I$ such that $xK = K$. Now the short exact sequence

$$0 \longrightarrow 0 :_K x \longrightarrow K \xrightarrow{.x} K \longrightarrow 0$$

gives rise to a long exact sequence

$$\cdots \to H^I_{i+1}(M, K) \xrightarrow{g_i} H^I_i(M, 0 :_K x) \xrightarrow{k_i} H^I_i(M, K) \xrightarrow{.x} H^I_i(M, K) \to \cdots .$$

It follows from the hypothesis that $H^I_i(M, 0 :_K x)$ satisfies the finite condition for co-associated primes for all $i < s-1$. Hence $H^I_{s-1}(M, 0 :_K x)/IH^I_{s-1}(M, 0 :_K x)$ also satisfies the finite condition for co-associated primes. The long exact sequence induces two short exact sequences

$$0 \longrightarrow \operatorname{Im} g_{s-1} \longrightarrow H^I_{s-1}(M, 0 :_K x) \longrightarrow \operatorname{Im} k_{s-1} \longrightarrow 0$$

and

$$H^I_s(M, K) \xrightarrow{.x} H^I_s(M, K) \longrightarrow \operatorname{Im} g_{s-1} \longrightarrow 0.$$

These short exact sequences induce the following exact sequences

$$\operatorname{Tor}_1(R/I, \operatorname{Im} k_{s-1}) \to \operatorname{Im} g_{s-1}/I\operatorname{Im} g_{s-1} \to H^I_{s-1}(M, 0 :_K x)/IH^I_{s-1}(M, 0 :_K x)$$

and

$$H^I_s(M, K)/IH^I_s(M, K) \xrightarrow{.x} H^I_s(M, K)/IH^I_s(M, K) \to \operatorname{Im} g_{s-1}/I\operatorname{Im} g_{s-1} \to 0.$$

As $x \in I$, the last exact sequence implies that

$$H^I_s(M, K)/IH^I_s(M, K) \cong \operatorname{Im} g_{s-1}/I\operatorname{Im} g_{s-1}.$$

Since $\operatorname{Tor}_1(R/I, \operatorname{Im} k_{s-1})$ and $H^I_{s-1}(M, 0 :_K x)/IH^I_{s-1}(M, 0 :_K x)$ satisfy the finite condition for co-associated primes, $\operatorname{Im} g_{s-1}/I\operatorname{Im} g_{s-1}$ also satisfies the finite condition for co-associated primes. Therefore $H^I_s(M, K)/IH^I_s(M, K)$ satisfies the finite condition for co-associated primes. It follows that

$$H^I_s(M, N)/IH^I_s(M, N)$$

satisfies the finite condition for co-associated primes. Finally, combining Lemma 2.7 with Lemma 2.8 we obtain

$$\begin{aligned}\operatorname{Coass}(H^I_s(M, N)) &= \operatorname{Coass}(H^I_s(M, N)) \cap V(I)\\ &= \operatorname{Coass}(H^I_s(M, N)/IH^I_s(M, N))\end{aligned}$$

which is a finite set. $\square$

The following theorem provides some equivalent properties of minimax local homology modules.

Theorem 3.6. *Let M be a finitely generated R-module and N a semi-discrete linearly compact R-module such that $N/(\bigcap_{t>0} I^t N)$ is an artinian R-module. Let s be a non-negative integer. Then the following statements are equivalent*:

(i) $H_i^I(M,N)$ *is an I-coartinian minimax R-module for all $i < s$*;
(ii) $H_i^I(M,N)$ *is a minimax R-module for all $i < s$*;
(iii) *There exists a positive integer t such that* $H_i^I(M,N)/0 :_{H_i^I(M,N)} I^t$ *is minimax for all $i < s$.*

Proof. (i)⇒(ii) and (ii)⇒(iii) are clear.

(ii)⇒(i) follows from Theorem 3.3.

(iii)⇒(ii) The proof is by induction on s. When $s = 0$, the implication is clear.

Let $s > 0$. From the inductive hypothesis $H_i^I(M,N)$ is minimax for all $i < s-1$. It remains to prove that $H_{s-1}^I(M,N)$ is minimax. It follows from [8, 2.1] that $I^t H_{s-1}^I(M,N)$ is minimax. Then there is a finitely generated submodule G of $I^t H_{s-1}^I(M,N)$ such that $I^t H_{s-1}^I(M,N)/G$ is artinian. We now have a short exact sequence

$$0 \longrightarrow I^t H_{s-1}^I(M,N)/G \longrightarrow H_{s-1}^I(M,N)/G \longrightarrow H_{s-1}^I(M,N)/I^t H_{s-1}^I(M,N) \longrightarrow 0.$$

Note that $H_{s-1}^{I^t}(M,N) \cong H_{s-1}^I(M,N)$. By Theorem 3.3

$$H_{s-1}^{I^t}(M,N)/I^t H_{s-1}^{I^t}(M,N)$$

is artinian. Hence $H_{s-1}^I(M,N)/I^t H_{s-1}^I(M,N)$ is artinian. The last exact sequence implies that $H_{s-1}^I(M,N)/G$ is artinian and then $H_{s-1}^I(M,N)$ is minimax. □

4. Applications to generalized local cohomology

In this section $(R,\mathfrak{m})$ will be a complete local noetherian ring with the maximal ideal $\mathfrak{m}$. Let M be an R-module and $E(R/\mathfrak{m})$ the injective envelope of $R/\mathfrak{m}$. The module $D(M) = \mathrm{Hom}(M, E(R/\mathfrak{m}))$ is called the Matlis dual of M. If M is a Hausdorff linearly topologized R-module, then the *Macdonald dual* of M is defined by $M^* = \mathrm{Hom}(M, E(R/\mathfrak{m}))$ the set of continuous homomorphisms of R-modules ([17, §9]). The topology on M^* is defined as in [17, 8.1]. Moreover, if M is semi-discrete, then the topology of M^* coincides with that induced on it as a submodule of $E(R/\mathfrak{m})^M$, where $E(R/\mathfrak{m})^M = \prod_{x\in M} (E(R/\mathfrak{m}))^x$, $(E(R/\mathfrak{m}))^x = E(R/\mathfrak{m})$ for all $x \in M$ ([17, 8.6]).

A Hausdorff linearly topologized R-module is $\mathfrak{m}$-*primary* if each element of M is annihilated by a power of $\mathfrak{m}$. A Hausdorff linearly topologized R-module M is *linearly discrete* if every $\mathfrak{m}$-primary quotient of M is discrete. It should be noted that if M is linearly discrete, then M is semi-discrete. The direct limit of a direct system of linearly discrete R-modules is linearly discrete ([17, 6.2, 6.7]).

Theorem 4.1 ([17, 5.8, 6.2, 9.3, 9.12, 9.13])**.**

(i) *A Hausdorff linearly topologized R-module M is semi-discrete if and only if $D(M) = M^*$.*
(ii) *If M is a linearly discrete R-module, then $D(M) = M^*$.*
(iii) *If M is linearly compact, then M^* is linearly discrete (hence semi-discrete). If M is semi-discrete, then M^* is linearly compact.*
(iv) *If M is linearly compact or linearly discrete, then we have a topological isomorphism $\omega : M \xrightarrow{\simeq} M^{**}$.*

Part (iii) of Theorem 4.1 follows from [17, Proof of 9.3, page 235].

Lemma 4.2 ([23, 4.6])**.** *Let M be a Hausdorff linearly topologized R-module.*

(i) *If M is I-coartinian, then M^* is I-cofinite.*
(ii) *If M is minimax, then M^* is also minimax.*

Lemma 4.3 ([6, 6.7])**.** *Let N be a finitely generated R-module and M a linearly compact R-module. Then*

$$(\mathrm{Tor}_i^R(N, M))^* \cong \mathrm{Ext}_R^i(N, M^*),$$
$$\mathrm{Tor}_i^R(N, M^*) \cong (\mathrm{Ext}_R^i(N, M))^*$$

for all $i \geq 0$.

It is well-known that the generalized local cohomology modules $H_I^i(M, N)$ of M, N was introduced by J. Herzog by defining

$$H_I^i(M, N) = \varinjlim_t \mathrm{Ext}_R^i(M/I^t M, N) \quad ([13], [26]).$$

Lemma 4.4 ([28, 3.4])**.** *Let M be a finitely generated R-module. If N is a semi-discrete linearly compact R-module, then $H_I^i(M, N)$ is a linearly discrete R-module for all $i \geq 0$.*

We have the duality theorem.

Theorem 4.5 ([22, 2.3 (ii)], [28, 4.5])**.** *Let M be a finitely generated R-module. Then*

(i) *For any R-module N and for all $i \geq 0$,*
$$H_i^I(M, D(N)) \cong D(H_I^i(M, N));$$
(ii) *If N is a linearly compact R-module, then for all $i \geq 0$,*
$$H_i^I(M, N^*) \cong (H_I^i(M, N))^*,$$
$$H_I^i(M, N^*) \cong (H_i^I(M, N))^*;$$
(iii) *If N is a semi-discrete linearly compact R-module, then we have topological isomorphisms of R-modules for all $i \geq 0$,*
$$H_I^i(M, N^*) \cong (H_i^I(M, N))^*,$$
$$H_i^I(M, N^*) \cong (H_I^i(M, N))^*.$$

Corollary 4.6. ([28, 4.6]) *Let M be a finitely generated R-module.*

(i) *If N is a linearly compact R-module, then for all $i \geq 0$,*

$$H_I^i(M,N) \cong (H_i^I(M,N^*))^*,$$
$$H_i^I(M,N) \cong (H_I^i(M,N^*))^*;$$

(ii) *If N is a semi-discrete linearly compact R-module, then we have topological isomorphisms of R-modules for all $i \geq 0$,*

$$H_I^i(M,N) \cong (H_i^I(M,N^*))^*,$$
$$H_i^I(M,N) \cong (H_I^i(M,N^*))^*.$$

The following Theorems 4.7 and 4.9 are inspired by A. Grothendieck's conjecture and C. Huneke's question.

Theorem 4.7. *Let M be a finitely generated R-modules and N a semi-discrete linearly compact R-module such that $\Gamma_I(N)$ is a finitely generated R-module. Let s be a non-negative integer. If $H_I^i(M,N)$ is minimax for all $i < s$, then $H_I^i(M,N)$ is I-cofinite for all $i < s$ and $0 :_{H_I^s(M,N)} I$ is a finitely generated R-module. In particular, $\operatorname{Ass}(H_I^s(M,N))$ is a finite set.*

Proof. From Theorem 4.5(ii) we have the following isomorphism for all $i \geq 0$,

$$H_i^I(M,N^*) \cong H_I^i(M,N)^*.$$

Since $H_I^i(M,N)$ is minimax for all $i < s$, $H_i^I(M,N^*)$ is also minimax for all $i < s$ by Lemma 4.2(ii). It should be noted by Theorem 4.1(iii) that N^* is a semi-discrete linearly compact R-module. Moreover, $\Lambda_I(N^*) \cong \Gamma_I(N)^*$ by [6, 6.4(ii)]. Then $\Lambda_I(N^*)$ is an artinian R-module. But $\Lambda_I(N^*) \cong N^*/(\bigcap\limits_{t>0} I^t N^*)$. It means that $N^*/(\bigcap\limits_{t>0} I^t N^*)$ is an artinian R-module. By virtue of Theorem 3.3, $H_i^I(M,N^*)$ is I-coartinian for all $i < s$ and $H_s^I(M,N^*)/IH_s^I(M,N^*)$ is an artinian R-module. We now consider the isomorphisms

$$H_I^i(M,N) \cong H_i^I(M,N^*)^*$$

by Corollary 4.6(i) and

$$(H_s^I(M,N^*)/IH_s^I(M,N^*))^* \cong 0 :_{H_s^I(M,N^*)^*} I \cong 0 :_{H_I^s(M,N)} I$$

by Lemma 4.3. Thus Lemma 4.2(i) provides that $H_I^i(M,N)$ is I-cofinite for all $i < s$ and $0 :_{H_I^s(M,N)} I$ is a finitely generated R-module. Note that for an R-module L, we have Bourbaki's result

$$\operatorname{Ass}\operatorname{Hom}(R/I,L) = V(I) \bigcap \operatorname{Ass}(L).$$

Then we get

$$\operatorname{Ass}(H_I^s(M,N)) = V(I) \bigcap \operatorname{Ass}(H_I^s(M,N)) = \operatorname{Ass}(0 :_{H_I^s(M,N)} I).$$

Finally, $\operatorname{Ass}(H_I^s(M,N))$ is a finite set. $\square$

Replacing the module M in Theorem 4.7 with the ring R, we have the following corollary. Note that there is a similar result in [2, 2.3], but the module N is finitely generated over a noetherian R.

Corollary 4.8. *Let N be a semi-discrete linearly compact R-module such that $\Gamma_I(N)$ is a finitely generated R-module. Let s be a non-negative integer. If $H_I^i(N)$ is minimax for all $i < s$, then $H_I^i(N)$ is I-cofinite for all $i < s$ and $0 :_{H_I^s(N)} I$ is a finitely generated R-module. In particular,* $\operatorname{Ass}(H_I^s(N))$ *is a finite set.*

Theorem 4.9. *Let M be a finitely generated R-modules and N a semi-discrete linearly compact R-module such that $\Gamma_I(N)$ is a finitely generated R-module. Let s be a non-negative integer. If $H_I^i(M,N)$ is minimax for all $i < s$ and G is a closed submodule of $H_I^s(M,N)$, then $0 :_{H_I^s(M,N)/G} I$ is finitely generated. In particular, the set* $\operatorname{Ass}(H_I^s(M,N)/G)$ *is a finite set.*

Proof. By Theorem 4.5(ii) we have the isomorphism

$$H_i^I(M,N^*) \cong H_I^i(M,N)^*$$

for all $i \geq 0$. Since $H_I^i(M,N)$ is minimax for all $i < s$, it follows from Lemma 4.2(ii) that $H_i^I(M,N^*)$ is also minimax for all $i < s$. It should be noted by Lemma 4.4 that since $H_I^s(M,N)$ is linearly discrete, so are G and $H_I^s(M,N)/G$. Set $K = H_I^s(M,N)/G$, we have the short exact sequence of linearly discrete R-modules with the continuous homomorphisms

$$0 \longrightarrow G \longrightarrow H_I^s(M,N) \longrightarrow K \longrightarrow 0.$$

It induces by Theorem 4.1(ii), (iii) a short exact sequence of linearly compact R-modules

$$0 \longrightarrow K^* \longrightarrow H_I^s(M,N)^* \longrightarrow G^* \longrightarrow 0$$

and the homomorphisms are continuous by [17, 8.2]. Furthermore, [17, 5.5] provides that these homomorphisms are open. Thus K^* can be considered as a closed submodule of $H_I^s(M,N)^*$ and then

$$G^* \cong H_I^s(M,N)^*/K^* \cong H_s^I(M,N^*)/K^*.$$

Hence $H_s^I(M,N^*)/K^*$ is minimax. On the other hand, from the proof of Theorem 4.7, N^* is a semi-discrete linearly compact R-module such that

$$N^*/(\bigcap_{t>0} I^t N^*)$$

is an artinian R-module. So K^*/IK^* is artinian by Theorem 3.4. Now combining Theorem 4.1 with Lemma 4.3 yields

$$(K^*/IK^*)^* \cong 0 :_{K^{**}} I \cong 0 :_K I.$$

Therefore $0 :_K I$ is finitely generated. In particular,

$$\operatorname{Ass}(K) = V(I) \bigcap \operatorname{Ass}(K) = \operatorname{Ass}(0 :_K I)$$

is a finite set. $\square$

Replacing the module M in Theorem 4.9 with the ring R, we have the following corollary. Note that there is a similar result in [2, 2.5], but the module N is finitely generated over a noetherian R.

Corollary 4.10. *Let N be a semi-discrete linearly compact R-module such that $\Gamma_I(N)$ is a finitely generated R-module. Let s be a non-negative integer. If $H^i_I(N)$ is minimax for all $i < s$ and G is a closed submodule of $H^s_I(N)$, then $0 :_{H^s_I(N)/G} I$ is finitely generated. In particular, the set $\mathrm{Ass}(H^s_I(N)/G)$ is a finite set.*

An R-module M is said to be weakly Laskerian if the set of associated primes of any quotient module of M is finite ([7]).

Lemma 4.11. *Let M be a linearly compact R-module. If M satisfies the finite condition for co-associated primes, then M^* is weakly Laskerian.*

Proof. It should be noted by Theorem 4.1 that M^* is a linearly discrete R-module and $M^{**} \cong M$. Let G be a submodule of M^*, the surjective homomorphism $M^* \to M^*/G$ induces an injective homomorphism $(M^*/G)^* \to M^{**} \cong M$. As M^* is a linearly discrete R-module, M^*/G is also linearly discrete and $(M^*/G)^* = D(M^*/G)$ by Theorem 4.1(ii). It follows from [27, 3.1(b)] that $\mathrm{Ass}(M^*/G) \subseteq \mathrm{Coass}(M^*/G)^*$. But $(M^*/G)^*$ can be considered as a submodule of M. Therefore $\mathrm{Coass}(M^*/G)^*$ is a finite set by the hypothesis and then $\mathrm{Ass}(M^*/G)$ is also a finite set. It means that M^* is weakly Laskerian. □

Theorem 4.12. *Let M be a finitely generated module and N a semi-discrete linearly compact R-module. Let s be a non-negative integer. If $H^i_I(M,N)^*$ satisfies the finite condition for co-associated primes for all $i < s$, then the module $0 :_{H^s_I(M,N)} I$ is weakly Laskerian. In particular, $\mathrm{Ass}(H^s_I(M,N))$ is a finite set.*

Proof. It follows from Theorem 4.1(iii) that N^* is a semi-discrete linearly compact R-module. By Theorem 4.5(ii) we have the following isomorphism for all $i \geq 0$,

$$H^I_i(M,N^*) \cong H^i_I(M,N)^*.$$

It follows that $H^I_i(M,N^*)$ satisfies the finite condition for co-associated primes for all $i < s$. Then $H^I_s(M,N^*)/IH^I_s(M,N^*)$ satisfies the finite condition for co-associated primes by Theorem 3.5. Now, combining Lemma 4.3 with Corollary 4.6(i) yields

$$(H^I_s(M,N^*)/IH^I_s(M,N^*))^* \cong 0 :_{H^I_s(M,N^*)^*} I \cong 0 :_{H^s_I(M,N)} I.$$

But $H^I_s(M,N^*)/IH^I_s(M,N^*)$ is linearly compact by Lemma 2.5 and Lemma 2.1. It follows from Lemma 4.11 that $0 :_{H^s_I(M,N)} I$ is weakly Laskerian and then $\mathrm{Ass}(0 :_{H^s_I(M,N)} I)$ is a finite set. Finally, we have

$$\mathrm{Ass}(H^s_I(M,N)) = V(I) \bigcap \mathrm{Ass}(H^s_I(M,N)) = \mathrm{Ass}(0 :_{H^s_I(M,N)} I)$$

which finishes the proof. □

Replacing the module M in Theorem 4.12 with the ring R, we have the following corollary.

Corollary 4.13. *Let N be a semi-discrete linearly compact R-module. Let s be a non-negative integer. If $H_I^i(N)^*$ satisfies the finite condition for co-associated primes for all $i < s$, then the module $0 :_{H_I^s(N)} I$ is weakly Laskerian. In particular,* $\operatorname{Ass}(H_I^s(N))$ *is a finite set.*

Theorem 4.14. *Let M be a finitely generated R-modules and N a semi-discrete linearly compact R-module such that $\Gamma_I(N)$ is a finitely generated R-module. Let s be a non-negative integer. Then the following statements are equivalent*:

(i) *$H_I^i(M,N)$ is an I-cofinite minimax R-module for all $i < s$*;
(ii) *$H_I^i(M,N)$ is a minimax R-module for all $i < s$*;
(iii) *There exists a positive integer t such that $I^t H_I^i(M,N)$ is minimax for all $i < s$*;

Proof. (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are clear.

(ii)$\Rightarrow$(i) By Theorem 4.5(ii) we have the isomorphism

$$H_i^I(M,N^*) \cong H_I^i(M,N)^*$$

for all $i \geq 0$. Since $H_I^i(M,N)$ is minimax for all $i < s$, it follows from Lemma 4.2(ii) that $H_i^I(M,N^*)$ is also minimax for all $i < s$. From the proof of Theorem 4.7 N^* is a semi-discrete linearly compact R-module and

$$N^*/(\bigcap_{t>0} I^t N^*)$$

is an artinian R-module. Then $H_i^I(M,N^*)$ is an I-coartinian minimax R-module for all $i < s$ by Theorem 3.6. On the other hand, Corollary 4.6 provides the isomorphism

$$H_I^i(M,N) \cong H_i^I(M,N^*)^*$$

for all $i \geq 0$. We conclude from Lemma 4.2 that $H_I^i(M,N)$ is an I-cofinite minimax R-module for all $i < s$.

(iii)$\Rightarrow$(ii) Assume that there exists a positive integer t such that $I^t H_I^i(M,N)$ is minimax for all $i < s$. We have the short exact sequence of Hausdorff linearly topologized R-modules

$$0 \longrightarrow I^t H_I^i(M,N) \longrightarrow H_I^i(M,N) \longrightarrow H_I^i(M,N)/I^t H_I^i(M,N) \longrightarrow 0,$$

in which the modules $H_I^i(M,N)$ and $H_I^i(M,N)/I^t H_I^i(M,N)$ are linearly discrete. The module $I^t H_I^i(M,N)$ is closed in $H_I^i(M,N)$ by [17, 6.2]. It should be noted by [17, 6.2] that if L is a linearly discrete R-module, then every homomorphism $M \to E(R/\mathfrak{m})$ is continuous. Then by an argument analogous to that used for the proof of [6, 6.5], we get a short exact sequence of R-modules

$$0 \longrightarrow (H_I^i(M,N)/I^t H_I^i(M,N))^* \longrightarrow H_I^i(M,N)^* \longrightarrow (I^t H_I^i(M,N))^* \longrightarrow 0.$$

Combining Lemma 4.3 with Theorem 4.5(ii) yields the following isomorphisms

$$H_i^I(M,N^*) \cong H_I^i(M,N)^* \text{ and}$$

$$(H_I^i(M,N)/I^tH_I^i(M,N))^* \cong 0 :_{H_I^i(M,N)^*} I^t \cong 0 :_{H_i^I(M,N^*)} I^t.$$

Thus the last short exact sequence induces an isomorphism

$$H_i^I(M,N^*)/0 :_{H_i^I(M,N^*)} I^t \cong (I^tH_I^i(M,N))^*.$$

As $I^tH_I^i(M,N)$ is minimax for all $i < s$, $H_i^I(M,N^*)/0 :_{H_i^I(M,N^*)} I^t$ is also minimax for all $i < s$ by Lemma 4.2(ii). From the proof above N^* is a semi-discrete linearly compact R-module such that $N^*/(\bigcap_{t>0} I^tN^*)$ is an artinian R-module. Hence $H_i^I(M,N^*)$ is a minimax R-module for all $i < s$ by Theorem 3.6. Finally, from the isomorphism $H_I^i(M,N) \cong H_i^I(M,N^*)^*$ we conclude that $H_I^i(M,N)$ is a minimax R-module for all $i < s$. □

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Yen Ngoc Do
Faculty of Mathematics & Computer Science
University of Science
VNU-HCM, Ho Chi Minh City, Vietnam
and
Faculty of Fundamental Science 2
Posts and Telecommunications Institute of Technology
Ho Chi Minh City, Vietnam
Email address: dongocyen.dhsp@gmail.com

Tri Minh Nguyen
Algebra and Geometry Research Group
Ton Duc Thang University
Ho Chi Minh City, Vietnam
and
Faculty of Mathematics and Statistics
Ton Duc Thang University
Ho Chi Minh City, Vietnam
Email address: nguyenminhtri@tdtu.edu.vn

Nam Tuan Tran
Department of Mathematics-Informatics
Ho Chi Minh University of Pedagogy
Ho Chi Minh city, Vietnam
Email address: namtuantran@gmail.com, namtt@hcmue.edu.vn